Holomorphic blocks for 3D non-Abelian partition functions†

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The pioneering work by Pestun[1] on the partition function of four-dimensional (4D) $\mathcal{N}=2$ theories has served as a trigger for great progress on localization computation of supersymmetric gauge theories in diverse dimensions and on various manifolds. The localization of three-dimensional (3D) theories is a recent focus of research. Kapustin, Willett, and Yaakov[2] extended Pestun's idea to gauge theories on S^3, and they obtained matrix model representations for the supersymmetric partition functions of these theories. We can solve these matrix-models in the large-N limit; for instance, the ABJM partition function was computed by Drukker, Marino, and Putrov[3]. They found that the free energy of the ABJM theory actually shows the $N^{3/2}$ scaling behavior, which had been suggested in the AdS/CFT argument. This result is a typical example of the power of the localization approach.

The efficiency of localization reaches beyond the large-N approximation. The matrix models for partition functions of $\mathcal{N}=2$ gauge theories on S^3 was derived in Ref.[4,5]. The integrant of this matrix model consists of a complicated combination of double-sine functions, and it appears difficult on first glance to evaluate it exactly. In Ref.[6], however, the authors succeessfully solved these matrix models exactly. In particular, the partition functions of 3D $\mathcal{N}=2$ $U(1)$ theories computed in Ref.[6] show the following factorization property:

$$Z^{U(1)}[S^3]=\sum_{i=1}^{N_f} Z_{\text{vort}}^{(i)}\,\widetilde{Z}_{\text{anti-vort}}^{(i)}. \tag{1}$$

Here, Z_{vort} and $\widetilde{Z}_{\text{anti-vort}}$ are the partition functions of the vortex and antivortex configurations on $S^1\times\mathrm{R}^2$ respectively. The summation is taken over the supersymmetric ground states that specify the vortex sector. This factorization into vortices is the 3D analogue of Pestun's expression,

$$Z^{U(1)}[S^4]=\int da\, Z_{\text{inst}}(a)\,\widetilde{Z}_{\text{anti-inst}}(a). \tag{2}$$

In this 4D case, ground states are labeled by the continuous moduli parameter a; therefore, we take the integral over it after combining the contributions from instantons and anti-instantons. The three-dimensional factorization is therefore expected to originate from the localization after changing the way of carrying out the localization computation.

In this paper, we extend this observation (1) to gauge theories with a more generic gauge group $SU(N)$. To compute the localization partition functions of these non-Abelian gauge theories, we need to evaluate the following complicated matrix integral:

$$Z=\frac{1}{N!}\int d^N x\; e^{-i\pi k\sum x_\alpha^2+2\pi i\xi\sum x_\alpha}\prod_{1\le\alpha<\beta\le N}$$
$$\times 4\sinh\pi b(x_\alpha-x_\beta)\,\sinh\pi b^{-1}(x_\alpha-x_\beta)$$
$$\times\prod_{\alpha=1}^{N}\prod_{i=1}^{N_f}\frac{s_b(x_\alpha+m_i+\mu_i/2+iQ/2)}{s_b(x_\alpha+m_i-\mu_i/2-iQ/2)}. \tag{3}$$

Here, $s_b(x)$ is the double-sine function[7]. Employing the Cauchy formula

$$\prod_{1\le\alpha<\beta\le N} 2\sinh(x_\alpha-x_\beta)2\sinh(\chi_\alpha-\chi_\beta)$$
$$=\sum_{\sigma\in S^N}(-1)^\sigma\prod_{\alpha}\prod_{\beta\ne\sigma(\alpha)}2\cosh(x_\alpha-\chi_\beta), \tag{4}$$

we succeeded to compute the matrix integral, and we found the following factorization:

$$Z^{SU(N)}[S^3]=\sum_{i_1}^{N_f}\cdots\sum_{i_N}^{N_f} Z_{\text{vort}}^{(\vec{i})}\,\widetilde{Z}_{\text{anti-vort}}^{(\vec{i})}, \tag{5}$$

where N_f is the number of flavors. This result suggests that the factorization is universal for the gauge theories in three dimensions, and we can expect a similar relation for gauge theories with other gauge groups on more generic three-dimensional manifolds. It would be also possible to re-derive our result physically without computing the partition functions explicitly.

References

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