Unitarity bounds from generalised Käbler identities

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A textbook result in Kähler geometry relates the de Rham with the Dolbeault Laplacian, $\Delta = 2\Delta_{\bar\partial}$. The topic of this note is a similar identity in the case of Sasaki-Einstein manifolds and its application in to the unitarity bounds in superconformal gauge theories (SCFTs):

$$\begin{aligned}\Delta &= 2\Delta_{\bar\partial_b} - \pounds_\eta^2 - 2\imath(n-d^0)\pounds_\eta + 2L\Lambda \\ &+ 2(n-d^0)L_\eta\Lambda_\eta + 2\imath(L_\eta\bar\partial_b^* - \bar\partial_b\Lambda_\eta).\end{aligned} \quad (1)$$

The right hand side features the tangential Cauchy-Riemann operator, the Lefschetz operator, and the action of the Reeb vector. The equation $\Delta = 2\Delta_{\bar\partial}$ can be derived from the Kähler identities, commutators between the Dolbeault and Lefschetz operators and their adjoints. The proof of equation 1 follows a similar route by obtaining Kähler-like identities that hold on Sasaki-Einstein manifolds. Those identities as well the details of the proof were worked out in[1)].

Equation 1 finds application in the AdS/CFT correspondence. Freund-Rubin compactification on Sasaki-Einstein manifolds yields supergravity duals of superconformal field theories. The AdS/CFT dictionary links the conformal energy of SCFT operators to the spectrum of Δ, their R-charge to that of the Lie-derivative along the Reeb vector, $\pounds_\eta$. The conformal energy, R-charge, and spin of any SCFT operator have to satisfy the unitarity bounds[4,5)], which should be reflected on the supergravity side in the spectrum of Δ. Indeed, it is possible to re-derive the unitarity bounds from supergravity when using equation 1 in conjunction with the calculations in[2,3)].

This leads us to the spectral problem for Δ. Decompose the cotangent bundle as $T^*S = D^* \oplus \eta = \Omega^{1,0} \oplus \Omega^{0,1} \oplus \eta$ and consider a k-form ω with $\pounds_\eta\omega = \imath q$, $q \geq 0$, and $d^0 \leq n$. Clearly all terms on the right hand side of 1 are positive definite except for the mixed term $M = \imath(L_\eta\bar\partial_b^* - \bar\partial_b\Lambda_\eta) = N + N^*$. M is self-adjoint and its spectrum is real. Moreover, $N^2 = 0$ and $N(\bigwedge^* D^*) \subset \bigwedge^* D^* \wedge \eta$ and $N(\bigwedge^* D^* \wedge \eta) = 0$. That is, N maps horizontal to vertical forms and annihilates the latter. N^* behaves accordingly and it follows that $\langle\omega, M\omega\rangle$ vanishes if ω is horizontal or vertical. This is also the case if ω is neither horizontal nor vertical yet holomorphic in the $\bar\partial_b$-sense. As long as we restrict to one of these cases, 1 takes the form of a bound on the spectrum of Δ.

This was conjectured and partially shown in the context of the calculations of the superconformal index in[2,3)]. Here, the spectrum was constructed from primitive elements of $\Omega^{p,q}$. For such forms, 1 clearly implies

$$\Delta \geq q^2 + 2q(n-d^0) \quad (2)$$

with equality if and only if $\bar\partial_b\omega = \bar\partial_b^*\omega = 0$. In the Kähler case, the latter of these is implied by transversality — $d^*\omega = 0$. Here however, $d^*\omega = 0$ leads only to the vanishing of the horizontal component of $\bar\partial_b^*\omega$. Indeed,

$$\partial_b^*\omega = \imath L_\eta\Lambda\omega, \quad \bar\partial_b^*\omega = -\imath L_\eta\Lambda\omega, \quad (3)$$

which vanishes since ω was assumed to be primitive. Assuming that every element of $H^{p,q}_{\bar\partial_b}(S)$ has a representative closed under $\bar\partial_b^*$, the bound 2 is saturated on the elements of $H^{p,q}_{\bar\partial_b}(S)$. These are the forms that correspond to the short multiplets in the SCFT, and 2 together with the expressions for the derived eigenmodes of Δ given in[2,3)] allows to recover the unitarity bounds from supergravity.

Since we found Sasaki-Einstein equivalents of both $\Delta = 2\Delta_{\bar\partial}$ and the Kähler identities, it is tempting to ask how much more of Kähler geometry can be generalized. For example, since $\Delta_{\bar\partial}$ is self-adjoint and elliptic, one can show that $\Omega^k_{\mathbb{C}} = \mathcal{H}^k \oplus \Delta_{\bar\partial}(\Omega^k_{\mathbb{C}})$ which implies Hodge's theorem. Similarly, the relation between the de Rham and Hodge Laplacians allows for an isomorphism between the respective spaces of harmonic forms. However, $\Delta_{\bar\partial_b}$ is not elliptic. Recall that $\Delta_{\bar\partial_b}$ is elliptic if the symbol $\sigma_{\Delta_{\bar\partial_b}} : Hom(\Omega^k_{\mathbb{C}}, \Omega^k_{\mathbb{C}}) \otimes S^2(T^*S)$ maps any non-zero $\omega \in T^*S$ to an automorphism on $\Omega^k_{\mathbb{C}}$. When calculating the symbol one essentially keeps only those terms of $\Delta_{\bar\partial_b}$ that are of highest order in derivatives. In the context of the tangential Cauchy-Riemann operator, this means that ∂_b and $\bar\partial_b$ can be taken to be anticommuting and that the overall result is essentially the same as for the symbol of the Dolbeault Laplacian on a Kähler manifold. It turns out, that $\sigma_{\Delta_{\bar\partial_b}}(\eta) = 0$ and $\Delta_{\bar\partial_b}$ is not elliptic, yet transversally elliptic.

An obvious problem of interest is the extension of the results presented here beyond the Sasaki-Einstein case. As long as there is a dual SCFT, there is a unitarity bound meaning that there should be some equivalent of 1.

References
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