Towards U($N|M$) knot invariant from ABJM theory

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The knot invariant can be realized using the Wilson loop operator in Chern–Simons gauge theory. Especially for the invariant for unknot and torus knot, there exists an integral representation analogous to the U(N) symmetric matrix model. Recently it was pointed out that the partition function of ABJM theory on S^3 can be written as a supermatrix integral.[1)] From this point of view, it is natural to explore a possibile connection between the ABJM Wilson loop and a knot invariant.

The ABJM partition function is written as follows,

$$\mathcal{Z} = \frac{1}{N!^2} \int [dx]^N [dy]^N \det_{1\le i,j\le N} \left(\frac{1}{2\cosh \frac{x_i - y_j}{2}} \right)^2 , \tag{1}$$

where $[dx] = \frac{dx}{2\pi} e^{-\frac{1}{2g_s}x^2}$ and $[dy] = \frac{dx}{2\pi} e^{\frac{1}{2g_s}y^2}$ with the string coupling constant $g_s = 2\pi i/k$. In this expression the Wilson loop operator in the representation R is given by the corresponding character of U($N|N$) group, $W_R \to \mathrm{Str}_R\, U(x;y)$ with the holonomy matrix $U(x;y) = \mathrm{diag}(e^{x_1}, \cdots, e^{x_N}, -e^{y_1}, \cdots, -e^{y_N})$. When the partition λ, corresponding to the representation R, satisfies $\Lambda_{N+1} > N$, this character is decomposed into that for SU(N) which is written in terms of the Schur function,

$$\mathrm{Str}_R\, U(x;y) = s_\mu(e^x)\, s_\nu(e^y) \prod_{i,j=1}^{N} (e^{x_i} - e^{y_j}) , \tag{2}$$

where $\mu_i^t = \lambda_{i+N}^t$ and $\nu_i^t = \lambda_{i+N}$. We consider this case in particular. Thus the integral representation for the unknot Wilson loop in ABJM theory is now written

$$\Big\langle W_R(K_{\mathrm{unknot}}) \Big\rangle = \int [dx]^N [dy]^N \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2}} \right) \prod_{i=1}^{N} e^{x_i \xi_i + y_i \eta_i} \tag{3}$$

with $\xi_i = \lambda_i - i + 1/2$, $\eta_i = \lambda_i^t - i + 1/2$. We can compute this integral by applying the Fourier transform formula $1/\cosh w = \int \frac{dz}{\pi} e^{2iwz/\pi}/\cosh z$,

$$\Big\langle W_R(K_{\mathrm{unknot}}) \Big\rangle = k^{-N} q^{\frac{1}{2}(C_2(\mu) - C_2(\nu))} \times \det_{1\le i,j\le N} \left(\frac{1}{q^{\frac{1}{2}(\xi_i+\eta_j)} + q^{-\frac{1}{2}(\xi_i+\eta_j)}} \right) , \tag{4}$$

where the parameter is defined as $q = e^{g_s}$ and $C_2(\lambda) = \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \frac{1}{2}\right)^2 - \left(-i + \frac{1}{2}\right)^2 \right)$ is the second Casimir operator, corresponding to the framing factor. This shows that the U($N|N$) character average is factorized into that for U(1|1) theory. This kind of property is called Giambelli compatibility.[2)]

In order to see the connection to the ordinary knot invariant in U(N) from the determinantal expression (4), it is convenient to rewrite as follows,

$$k^{-N} q^{\frac{1}{2}(C_2(\mu) - C_2(\nu))} \prod_{i,j=1}^{N} \left(q^{\frac{1}{2}(\xi_i+\eta_j)} + q^{-\frac{1}{2}(\xi_i+\eta_j)} \right)^{-1} \times \prod_{i<j}^{N} \left(q^{\frac{1}{2}(\xi_i-\xi_j)} - q^{-\frac{1}{2}(\xi_i-\xi_j)} \right) \left(q^{\frac{1}{2}(\eta_i-\eta_j)} - q^{-\frac{1}{2}(\eta_i-\eta_j)} \right) . \tag{5}$$

The last two factors coincide with the Wilson loop average in U(N) theory, which is given by the quantum dimension of the representation R, up to the normalization constant.

The integral formula shown above can be generalized to the situation for the torus knot, which is labeled by two coprime integers (P, Q). In this case the partition function is slightly modified

$$\mathcal{Z}_{(P,Q)} = \frac{1}{N!^2} \int [dx]^N [dy]^N \times \det_{1\le i,j\le N} \left(\frac{1}{2\cosh \frac{x_i - y_j}{2P}} \right) \det_{1\le i,j\le N} \left(\frac{1}{2\cosh \frac{x_i - y_j}{2Q}} \right) . \tag{6}$$

We can show this torus knot partition function is just given by the symplectic transform of the original unknot partition function. Thus it can be shown that they are related in a simple way, $\mathcal{Z}_{(P,Q)} = (PQ)^N \mathcal{Z}_{(1,1)}$. Since there exists the U($N|N$) character, written in terms of the Schur function, $s_\lambda(u^Q; v^Q) = \sum_\mu c^\mu_{\lambda,Q} s_\mu(u;v)$, as well as the ordinary U(N) theory, finally the torus knot Wilson loop average can be expressed as a linear combination of that for the fractionally framed unknot,

$$\Big\langle W_R(K_{P,Q}) \Big\rangle = \sum_V c^V_{R,Q} \Big\langle W_R(K_{1,f}) \Big\rangle \tag{7}$$

with the framing number $f = Q/P$. This is just a supersymmetric version of the Rosso–Jones formula for the torus knot invariant.[3)]

References
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