Twelfth-order QED contributions to the muon $g-2$

R. Yamazaki[*1,*2] and M. Nio[*2,*1]

The anomalous magnetic moment of the muon, $a_\mu \equiv (g_\mu - 2)/2$, has occupied a central role in testing the validity of the standard model of elementary particles (SM). The new measurement of a_μ at Fermilab[1] is consistent with that at Brookhaven,[2] and the long-standing tension between experiment and theory remains unresolved. The average of the two experiments and the SM prediction[3] are given as

$$a_\mu(\text{Exp}) = 116\ 592\ 061\ (41) \times 10^{-11}, \quad (1)$$
$$a_\mu(\text{SM}) = 116\ 591\ 810\ (43) \times 10^{-11}, \quad (2)$$

respectively, and the difference is $(251 \pm 59) \times 10^{-11}$ corresponding to 4.2 σ.

The theoretical prediction of a_μ has been calculated by considering all three forces of SM. The contribution from the quantum electrodynamics (QED) is dominant and has been determined up to the tenth order of the perturbation theory. It has been considered sufficiently well known as its uncertainty is 1.0×10^{-12}. The assigned uncertainty was derived from the rough estimate of the leading-order contribution of the twelfth-order term. Recently, we have calculated, not guessed, the two types of Feynman diagrams shown in Fig. 1 that are supposed to give the leading contributions in the twelfth order.

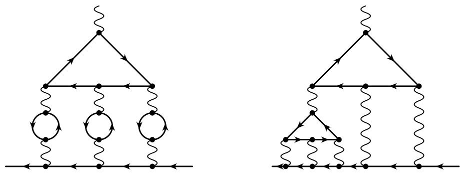

Fig. 1. Some Feynman vertex diagrams of the twelfth-order QED containing light-by-light scattering subdiagrams.

Because the muon is about two hundred times heavier than the electron, enhancement factors arise in some QED contributions to a_μ. One famous origin of the enhancement is a light-by-light scattering vertex diagram (LL6), in which an external magnetic photon is attached to an electron loop. Another is a vacuum-polarization subdiagram (VP) consisting of an electron loop. The left diagram of Fig. 1 is a combination of two enhancement mechanisms of the twelfth order of the QED perturbation theory. We have calculated its numerical contribution taking into account all possible insertions of three second-order VPs (P2s) to LL6. In addition, diagrams with insertions of one fourth-order VP (P4) and two P2s to LL6 were calculated. The coefficients of $(\alpha/\pi)^6$, where α is the fine-structure constant, are obtained as follows:

$$a_\mu^{(12)}[\text{LL6}^{(e)}\text{\_P2}^{(e)}\text{P2}^{(e)}\text{P2}^{(e)}] = 2415.256\ (53), \quad (3)$$
$$a_\mu^{(12)}[\text{LL6}^{(e)}\text{\_P2}^{(\mu)}\text{P2}^{(e)}\text{P2}^{(e)}] = 367.974\ (15), \quad (4)$$
$$a_\mu^{(12)}[\text{LL6}^{(e)}\text{\_P4}^{(e)}\text{P2}^{(e)}] = 1451.106\ (91), \quad (5)$$

where a superscript (e) or (μ) of LL6, P2, or P4 indicates a species of its fermion loop. The sum of the three gives the leading contribution of the twelfth-order QED:

$$\left(\frac{\alpha}{\pi}\right)^6 \times 4230 = 0.665 \times 10^{-12}\,. \quad (6)$$

It is within the uncertainty assigned to the QED contribution to a_μ.

A new enhancement mechanism appears at the twelfth order. The right diagram of Fig. 1 (LL6_LL6) contains a "child" LL6 as a subdiagram. When the mass of an external fermion is much heavier than that of a loop fermion, the slope of F_1 form factor of LL6 has a large enhancement factor[4]

$$m_\mu^2 \frac{dF_1(q^2)}{dq^2}\bigg|_{q^2=0} \propto \left(\frac{m_\mu}{m_e}\right)^2 \sim 40000\,. \quad (7)$$

It is uncertain that such a critical enhancement factor exists in the LL6_LL6 contribution to a_μ. Therefore, we calculated LL6_LL6 without any approximation except for numerical integration. The 360 Feynman vertex diagrams of the gauge-invariant set LL6_LL6 can be reduced to sixteen independent integrals. Each of the integrands consists of about 50 MByte text files. In principle, no renormalization is required for LL6_LL6. However, to conduct numerical integration, an integrand needs ultra-violet (UV) counter terms that cancel the UV divergences of a child LL6 vertex subdiagram and a light-by-light scattering subdiagram (LL). The UV counter terms we constructed are the form factor $F_1(0)$ for a child LL6 and the light-by-light scattering tensor $\Pi^{\mu\nu\rho\sigma}(0,0,0,0)$ for a LL. The sum of the UV counter terms exactly vanishes when all gauge-invariant diagrams are summed up. The numerical calculation has been performed, and the result will be reported elsewhere.

Numerical calculations were conducted on RIKEN's supercomputer HOKUSAI BigWaterfall. This work was partly supported by KAKENHI 16K05338, 20H05646, and 22K03646.

References

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*1 Department of Physics, Saitama University
*2 RIKEN Nishina Center