Formalism of CRDW

February 15, 2018

Munetake Ichimura

RIKEN Nishina Center
e-mail: ichimura@riken.jp

Contents

1 Preface

This manuscript presents the formalism used in the program CRDW, which calculates the nucleon induced inelastic and charge exchange reactions by means of a distorted wave impulse approximation (DWIA) and a continuum RPA (CRPA). CRDW means Continuum RPA + DWIA.

We consider the nucleon-nucleus (NA) reaction

$$\mathrm{A}(\mathrm{N},\mathrm{N}')\mathrm{X}_n \tag{1.1}$$

where N and N′ represent the neutron (n) or the proton (p), and A does the target nucleus. X denotes the residual nuclear system, which can be continuum states and can include the Δ isobar. The suffix n denotes the n-th excited state of the system X. We restrict the target A to a doubly closed shell, thus its total spin $J_\mathrm{A} = 0$.

The present formalism is based on refs.[1]-[5] with some changes of notations.

2 Kinematics

2.1 Notations

We use the following notations for the kinematical variables of the NA reactions,

(1) Masses and Charges

		(invariant) masses	charges
Incident nucleon	N	m_N	Z_N
Outgoing nucleon	N′	$m_{\mathrm{N}'}$	$Z_{\mathrm{N}'}$
Target	A	m_A	Z_A
Residual nuclear system	X_n	m_X^n	Z_X

(2) Momenta and energies in the laboratory (lab.) and the center of mass (c.m.) systems

		lab.	c.m.
Incident nucleon kinetic energy		K_lab	—
Scattering angle		θ_lab	θ_cm
Momenta			
Incident nucleon	N	$\boldsymbol{k}_{i,\mathrm{lab}}$	$\boldsymbol{k}_i$
Outgoing nucleon	N′	$\boldsymbol{k}_{f,\mathrm{lab}}$	$\boldsymbol{k}_f$
Target	A	$\boldsymbol{k}_{\mathrm{A},\mathrm{lab}}$	$\boldsymbol{k}_\mathrm{A}$
Residual nuclear system	X_n	$\boldsymbol{k}_{\mathrm{X},\mathrm{lab}}^n$	$\boldsymbol{k}_\mathrm{X}^n$
Energies			
Incident nucleon	N	$E_{\mathrm{N},\mathrm{lab}}$	E_N
Outgoing nucleon	N′	$E_{\mathrm{N}',\mathrm{lab}}$	$E_{\mathrm{N}'}$
Target	A	$E_{\mathrm{A},\mathrm{lab}}$	E_A
Residual nuclear system	X_n	$E_{\mathrm{X},\mathrm{lab}}^n$	E_X^n
Momentum transfer from N to N′		$\boldsymbol{q}_\mathrm{lab}$	$\boldsymbol{q}_\mathrm{cm}$
Energy transfer from N to N′		$-\omega_\mathrm{lab}$	$-\omega_\mathrm{cm}$

(3) Invariants

Mandelstam variable s	s_NA
Mandelstam variable t	t_NA

(4) Relative motion

We treat the relative motion between N and A (N′ and X) in a non-relativistic way with the reduced energy prescription.

	initial	final
Reduced energy	μ_i	μ_f
Kinetic energy	K_i	K_f
Sommerfeld parameter	η_i	η_f

(5) Excitation energy with respect to the target ground state $\qquad \omega_n = m_{\mathrm{X}}^n - m_{\mathrm{A}}$

2.2 Formulas for the kinematical variables

Input kinematical variables are

$$m_{\mathrm{N}},\ m_{\mathrm{N}'},\ m_{\mathrm{A}},\ Z_{\mathrm{N}},\ Z_{\mathrm{N}'},\ Z_{\mathrm{A}},\ K_{\mathrm{lab}},\ \theta_{\mathrm{lab}},\ \omega_{\mathrm{lab}},$$

from which we calculate other kinematic variables by the following relations.

(1) **Incident channel**

$$E_{\mathrm{N,lab}} = m_{\mathrm{N}} + K_{\mathrm{lab}}, \qquad k_{i,\mathrm{lab}} = \sqrt{E_{\mathrm{N,lab}}^2 - m_{\mathrm{N}}^2} \tag{2.2}$$

$$E_{\mathrm{A,lab}} = m_{\mathrm{A}} \qquad k_{\mathrm{A,lab}} = 0 \tag{2.3}$$

$$s_{\mathrm{NA}} = (E_{\mathrm{N,lab}} + E_{\mathrm{A,lab}})^2 - k_{i,\mathrm{lab}}^2 = (m_{\mathrm{A}} + m_{\mathrm{N}})^2 + 2m_{\mathrm{A}}K_{\mathrm{lab}} \tag{2.4}$$

$$E_{\mathrm{N}} = \frac{s_{\mathrm{NA}} + m_{\mathrm{N}}^2 - m_{\mathrm{A}}^2}{2\sqrt{s_{\mathrm{NA}}}}, \qquad k_i = \sqrt{E_{\mathrm{N}}^2 - m_{\mathrm{N}}^2} \tag{2.5}$$

$$E_{\mathrm{A}} = \sqrt{s_{\mathrm{NA}}} - E_{\mathrm{N}}, \qquad \boldsymbol{k}_{\mathrm{A}} = -\boldsymbol{k}_i \tag{2.6}$$

(2) **Exit channel**

$$E_{\mathrm{N}',\mathrm{lab}} = E_{\mathrm{N,lab}} - \omega_{\mathrm{lab}}, \quad k_{f,\mathrm{lab}} = \sqrt{E_{\mathrm{N}',\mathrm{lab}}^2 - m_{\mathrm{N}'}^2} \tag{2.7}$$

$$\boldsymbol{q}_{\mathrm{lab}} = \boldsymbol{k}_{f,\mathrm{lab}} - \boldsymbol{k}_{i,\mathrm{lab}}, \qquad q_{\mathrm{lab}} = \sqrt{k_{i,\mathrm{lab}}^2 + k_{f,\mathrm{lab}}^2 - 2k_{i,\mathrm{lab}}k_{f,\mathrm{lab}}\cos\theta_{\mathrm{lab}}} \tag{2.8}$$

$$E_{\mathrm{X,lab}}^n = m_{\mathrm{A}} + \omega_{\mathrm{lab}}, \qquad \boldsymbol{k}_{\mathrm{X,lab}}^n = -\boldsymbol{q}_{\mathrm{lab}}, \qquad m_{\mathrm{X}}^n = \sqrt{\left(E_{\mathrm{X,lab}}^n\right)^2 - q_{\mathrm{lab}}^2} \tag{2.9}$$

$$E_{\mathrm{N}'} = \frac{s_{\mathrm{NA}} + m_{\mathrm{N}'}^2 - (m_{\mathrm{X}}^n)^2}{2\sqrt{s_{\mathrm{NA}}}}, \qquad k_f = \sqrt{E_{\mathrm{N}'}^2 - m_{\mathrm{N}'}^2} \tag{2.10}$$

$$\omega_{\mathrm{cm}} = E_{\mathrm{N}} - E_{\mathrm{N}'}, \qquad E_{\mathrm{X}}^n = E_{\mathrm{A}} + \omega_{\mathrm{cm}}, \qquad \boldsymbol{k}_{\mathrm{X}} = -\boldsymbol{k}_f \tag{2.11}$$

$$t_{\mathrm{NA}} = \omega_{\mathrm{lab}}^2 - q_{\mathrm{lab}}^2, \qquad \boldsymbol{q}_{\mathrm{cm}} = \boldsymbol{k}_f - \boldsymbol{k}_i, \qquad q_{\mathrm{cm}} = \sqrt{\omega_{\mathrm{cm}}^2 - t_{\mathrm{NA}}} \tag{2.12}$$

$$\theta_{\mathrm{cm}} = \sin^{-1}\left(\frac{k_{\mathrm{lab},f}}{k_f}\sin\theta_{\mathrm{lab}}\right) \tag{2.13}$$

(3) **Relative motion**

$$\mu_i = \frac{E_{\mathrm{N}}E_{\mathrm{A}}}{\sqrt{s_{\mathrm{NA}}}}, \qquad K_i = \frac{k_i^2}{2\mu_i}, \qquad \eta_i = Z_{\mathrm{N}}Z_{\mathrm{A}}\alpha\frac{\mu_i}{k_i} \tag{2.14}$$

$$\mu_f = \frac{E_{\mathrm{N}'}\left(E_A + \omega_{\mathrm{cm}}\right)}{\sqrt{s_{\mathrm{NA}}}}, \qquad K_f = \frac{k_f^2}{2\mu_f}, \qquad \eta_f = Z_{\mathrm{N}'}Z_{\mathrm{X}}\alpha\frac{\mu_f}{k_f} \tag{2.15}$$

with the fine-structure constant α.

2.3 Coordinate systems for the nucleon-nucleus system

The following coordinate systems are used for the NA system.

(1) **The $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ system**

This is defined as

$$\hat{\boldsymbol{z}} = \frac{\boldsymbol{k}_{i,\mathrm{lab}}}{|\boldsymbol{k}_{i,\mathrm{lab}}|} = \frac{\boldsymbol{k}_i}{|\boldsymbol{k}_i|}, \qquad \hat{\boldsymbol{y}} = \frac{\boldsymbol{k}_{i,\mathrm{lab}} \times \boldsymbol{k}_{f,\mathrm{lab}}}{|\boldsymbol{k}_{i,\mathrm{lab}} \times \boldsymbol{k}_{f,\mathrm{lab}}|} = \frac{\boldsymbol{k}_i \times \boldsymbol{k}_f}{|\boldsymbol{k}_i \times \boldsymbol{k}_f|}, \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} , \tag{2.16}$$

Their directions are denoted by (x, y, z). This is used in the calculation.

(2) **The $[\boldsymbol{S}, \boldsymbol{N}, \boldsymbol{L}]$, and $[\boldsymbol{S}', \boldsymbol{N}', \boldsymbol{L}']$ systems**

They are defined as

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{N}} = \hat{\boldsymbol{y}}, \quad \hat{\boldsymbol{L}} = \hat{\boldsymbol{z}} \tag{2.17}$$

and

$$\hat{\boldsymbol{N}}' = \hat{\boldsymbol{N}} \ , \quad \hat{\boldsymbol{L}}' = \frac{\boldsymbol{k}_{f,\mathrm{lab}}}{|\boldsymbol{k}_{f,\mathrm{lab}}|} \ , \quad \hat{\boldsymbol{S}}' = \hat{\boldsymbol{N}}' \times \hat{\boldsymbol{L}}' \ . \tag{2.18}$$

Their directions are denoted by S, N, L and S', N', L', respectively. The measured quantities in the lab. system are usually presented in these coordinate systems.

(3) **The $[\boldsymbol{q}, \boldsymbol{n}, \boldsymbol{p}]$ system**

This is defined as

$$\hat{\boldsymbol{q}} = \frac{\boldsymbol{q}_{\mathrm{cm}}}{|\ \boldsymbol{q}_{\mathrm{cm}}\ |} \ , \quad \hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}, \quad \hat{\boldsymbol{p}} = \hat{\boldsymbol{q}} \times \hat{\boldsymbol{n}} \ , \tag{2.19}$$

Their directions are denoted by (q, n, p), respectively. We note that the suffix 'cm' is omitted for $\hat{\boldsymbol{q}}$ following the usual convention, but must be distinguished from $\boldsymbol{q}$ defined eq. (5.1). This system is often used to specify the c.m. quantities.

3 General formulas for the observables

3.1 T-matrix

The T-matrix elements for the reaction (1.1) in the NA c.m. system is specified as

$$\langle \boldsymbol{k}_f m_{s_f} \mathrm{N}', \Psi^n_{\mathrm{X}} | T | \boldsymbol{k}_i m_{s_i} \mathrm{N}, \Psi^0_{\mathrm{A}} \rangle \tag{3.20}$$

where m_{s_i} and m_{s_f} denote the spin projections of N and N′, and Ψ^0_{A} and Ψ^n_{X} are the wave function of the ground state of A and that of the n-th state of X, respectively. They are the eigenstates of the intrinsic Hamiltonian H_A of the A(X) system with the intrinsic energy $\mathcal{E}^0_{\mathrm{A}}$ and $\mathcal{E}^n_{\mathrm{X}}$, respectively,

$$H_A \Psi^0_{\mathrm{A}} = \mathcal{E}^0_{\mathrm{A}} \Psi^0_{\mathrm{A}}, \quad H_A \Psi^n_{\mathrm{X}} = \mathcal{E}^n_{\mathrm{X}} \Psi^n_{\mathrm{X}} \tag{3.21}$$

We take the approximation

$$\mathcal{E}^0_{\mathrm{A}} = m_{\mathrm{A}}, \qquad \mathcal{E}^n_{\mathrm{X}} = m^n_{\mathrm{X}} \tag{3.22}$$

3.2 Unpolarized double differential cross section

(1) **The center of mass system**

The unpolarized double differential cross section in the c.m. system is expressed as

$$I_{\rm cm}(\theta_{\rm cm},\omega_{\rm cm}) = \frac{d^2\sigma}{d\Omega_{\rm cm}d\omega_{\rm cm}}$$

$$= \frac{\mu_i\mu_f}{(2\pi)^2}\frac{k_f}{k_i}\frac{1}{2}\sum_{m_{s_i}m_{s_f}}\sum_n \left|\langle \boldsymbol{k}_f m_{s_f}{\rm N}', \Psi^n_{\rm X}|T|\boldsymbol{k}_i m_{s_i}{\rm N}, \Psi^0_{\rm A}\rangle\right|^2 \delta\left(\omega_{\rm cm} - (E^n_{\rm X} - E_{\rm A})\right) \quad (3.23)$$

To use a response function method, we introduce the cross section per unit excitation energy as

$$\frac{d^2\sigma}{d\Omega_{\rm cm}d\omega} = \frac{\mu_i\mu_f}{(2\pi)^2}\frac{k_f}{k_i}\frac{1}{2}\sum_{m_{s_i}m_{s_f}}\sum_n \left|\langle \boldsymbol{k}_f m_{s_f}{\rm N}', \Psi^n_{\rm X}|T|\boldsymbol{k}_i m_{s_i}{\rm N}, \Psi^0_{\rm A}\rangle\right|^2 \delta(\omega - \omega_n) \quad (3.24)$$

with

$$\omega_n = \mathcal{E}^n_{\rm X} - \mathcal{E}^0_{\rm A} = m^n_{\rm X} - m_{\rm A} \quad (3.25)$$

Then, the unpolarized double differential cross section in the c.m. system is given as

$$I_{\rm cm}(\theta_{\rm cm},\omega_{\rm cm}) = \frac{K}{2}\sum_{m_{s_f},m_{s_i}}\sum_n \left|\langle \boldsymbol{k}_f m_{s_f}{\rm N}', \Psi^n_{\rm X}|T|\boldsymbol{k}_i m_{s_i}{\rm N}, \Psi^0_{\rm A}\rangle\right|^2 \delta\left(\omega - \omega_n\right)) \quad (3.26)$$

with the kinematical factor

$$K = \frac{\mu_i\mu_f}{(2\pi)^2}\frac{k_f}{k_i}\frac{d\omega_n}{d\omega_{\rm cm}} = \frac{\mu_i\mu_f}{(2\pi)^2}\frac{k_f}{k_i}\frac{\sqrt{s_{\rm NA}}}{m^n_{\rm X}} \quad (3.27)$$

where we used eq.(2.10) to get

$$\frac{d\omega_n}{d\omega_{\rm cm}} = -\left(\frac{dE_{\rm N'}}{dm^n_{\rm X}}\right)^{-1} = \frac{\sqrt{s_{\rm NA}}}{m^n_{\rm X}} \quad (3.28)$$

(2) **The laboratory system**

The unpolarized double differential cross section in the lab. frame is obtained as

$$I_{\rm lab}(\theta_{\rm lab},\omega_{\rm lab}) = \frac{\partial(\Omega_{\rm cm},\omega_{\rm cm})}{\partial(\Omega_{\rm lab},\omega_{\rm lab})} I_{\rm cm}(\theta_{\rm cm},\omega_{\rm cm}) = \frac{k_{{\rm lab},f}}{k_f} I_{\rm cm}(\theta_{\rm cm},\omega_{\rm cm}) \quad (3.29)$$

The Jacobian is given in eq.(5-22) of ref. [6].

3.3 Spin observables

We introduce the notations, Tr, for the trace over the nucleon spin projections, e.g.

$$\mathrm{Tr}[AB] = \sum_{m_s m'_s}\langle m_s|A|m'_s\rangle\langle m'_s|B|m_s\rangle \quad (3.30)$$

and Tr$'$ for the sum over the final nuclear states $\Psi^n_{\rm X}$ with the energy restriction

$$\mathrm{Tr}'\left[TT^\dagger\right] \equiv \sum_n \langle\Psi^n_{\rm X}|T|\Psi^0_{\rm A}\rangle\langle\Psi^0_{\rm A}|T^\dagger|\Psi^n_{\rm X}\rangle\delta\left(\omega - \omega_n\right) \quad (3.31)$$

3.3.1 Polarization, analyzing power and polarization transfer coefficients

The polarization P_y, the analyzing power A_y and the polarization transfer coefficients D_{ij} are given by

$$P_y = \frac{\mathrm{Tr}\mathrm{Tr}'[TT^\dagger\sigma_{y,0}]}{\mathrm{Tr}\mathrm{Tr}'[TT^\dagger]} \ , \quad A_y = \frac{\mathrm{Tr}\mathrm{Tr}'[T\sigma_{y,0}T^\dagger]}{\mathrm{Tr}\mathrm{Tr}'[TT^\dagger]} \ , \quad D_{ij} = \frac{\mathrm{Tr}\mathrm{Tr}'[T\sigma_{j,0}T^\dagger\sigma_{i,0}]}{\mathrm{Tr}\mathrm{Tr}'[TT^\dagger]} \ . \tag{3.32}$$

where $\sigma_{i,0}$ is the Pauli spin matrix in the direction of i of the particle 0 (= the incident (exit) nucleon) .

The program CRDW first calculates D_{ij} $(i, j = x, y, z)$, and transforms them to D_{ij} $(i, j = q, n, p)$ by the relations

$$D_{nn} = D_{yy} \tag{3.33}$$

$$\begin{pmatrix} D_{qq} & D_{qp} \\ D_{pq} & D_{pp} \end{pmatrix} = \begin{pmatrix} \cos\theta_p & \sin\theta_p \\ -\sin\theta_p & \cos\theta_p \end{pmatrix} \begin{pmatrix} D_{xx} & D_{xz} \\ D_{zx} & D_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{pmatrix} \tag{3.34}$$

where θ_p is the angle between p and z directions.

Experimental data of D_{ij} are usually presented in the lab. system with respect to the $[\boldsymbol{S}, \boldsymbol{N}, \boldsymbol{L}]$ frame in the initial channel and the $[\boldsymbol{S}', \boldsymbol{N}', \boldsymbol{L}']$ frame in the final channel. namely, $D_{NN'}, D_{SS'}, D_{SL'}, D_{LS'}, D_{LL'}$. They are calculated by the relations

$$D_{NN'} = D_{yy} \tag{3.35}$$

$$\begin{pmatrix} D_{SS'} & D_{SL'} \\ D_{LS'} & D_{LL'} \end{pmatrix} = \begin{pmatrix} D_{xx} & D_{xz} \\ D_{zx} & D_{zz} \end{pmatrix} \begin{pmatrix} \cos(\theta_{\mathrm{lab}} + \Omega) & \sin(\theta_{\mathrm{lab}} + \Omega) \\ -\sin(\theta_{\mathrm{lab}} + \Omega) & \cos(\theta_{\mathrm{lab}} + \Omega) \end{pmatrix} \tag{3.36}$$

where Ω is the relativistic spin rotation angle. The present version of CRDW sets $\Omega = 0$. (Validity of this approximation is discussed in ref.[4].)

3.3.2 Polarized cross sections

For physical discussions, we decompose the NA T-matrix into

$$T = T_0 + T_n\sigma_{n,0} + T_q\sigma_{q,0} + T_p\sigma_{p,0} \ , \tag{3.37}$$

and introduce the polarized cross sections $I_{\mathrm{cm}}D_i$, which exclusively extract T_i as

$$\begin{aligned} I_{\mathrm{cm}}D_0 &= \frac{I_{\mathrm{cm}}}{4}[1 + D_{nn} + D_{qq} + D_{pp}] = K\mathrm{Tr}'[T_0T_0^\dagger] \ , \\ I_{\mathrm{cm}}D_n &= \frac{I_{\mathrm{cm}}}{4}[1 + D_{nn} - D_{qq} - D_{pp}] = K\mathrm{Tr}'[T_nT_n^\dagger] \ , \\ I_{\mathrm{cm}}D_q &= \frac{I_{\mathrm{cm}}}{4}[1 - D_{nn} + D_{qq} - D_{pp}] = K\mathrm{Tr}'[T_qT_q^\dagger] \ , \\ I_{\mathrm{cm}}D_p &= \frac{I_{\mathrm{cm}}}{4}[1 - D_{nn} - D_{qq} + D_{pp}] = K\mathrm{Tr}'[T_pT_p^\dagger] \ , \end{aligned} \tag{3.38}$$

Note the relation

$$D_0 + D_n + D_q + D_p = 1 \tag{3.39}$$

4 DWIA

4.1 T-matrix in DWIA

In DWIA the T-matrix of eq.(3.20) is given by

$$T^{fi}_{n0} = \langle \boldsymbol{k}_f m_{s_f} \mathrm{N}', \Psi^n_{\mathrm{X}} | T^{\mathrm{DWIA}} | \boldsymbol{k}_i m_{s_i} \mathrm{N}, \Psi^0_{\mathrm{A}} \rangle = \langle \chi_f \Psi^n_{\mathrm{X}} | \sum_k t^{(0)}_k | \Psi^0_{\mathrm{A}} \chi_i \rangle \tag{4.1}$$

where χ_i and χ_f are the distorted waves in the incident and the exit nucleons, respectively. The subscript i and f represent the sets of the variables $(\boldsymbol{k}_i m_{s_i} \mathrm{N})$ and $(\boldsymbol{k}_f m_{s_f} \mathrm{N}')$, respectively. $t^{(0)}_k$ is the free nucleon-nucleon (NN) t-matrix for the scattering of the incident nucleon with the k-th nucleon in the nucleus. It will be generalized to allow the Δ excitation or deexcitation.

In the momentum representation, it is written as

$$\begin{aligned} T^{fi}_{n0} &= \sum_k \int \frac{d\boldsymbol{k}'^3}{(2\pi)^3} \frac{d\boldsymbol{k}^3}{(2\pi)^3} \frac{d\boldsymbol{p}'^3_k}{(2\pi)^3} \frac{d\boldsymbol{p}^3_k}{(2\pi)^3} \Pi_{j\neq k} \frac{d\boldsymbol{p}^3_j}{(2\pi)^3}\ (2\pi)^3 \delta(\boldsymbol{k}' + \boldsymbol{p}'_k - \boldsymbol{k} - \boldsymbol{p}_k) \\ &\times\ \tilde{\chi}^\dagger_f(\boldsymbol{k}') \tilde{\Psi}^{n\dagger}_{\mathrm{X}}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}'_k, \cdots) \langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\mathrm{NA}} \tilde{\Psi}^0_{\mathrm{A}}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_k, \cdots) \tilde{\chi}_i(\boldsymbol{k}) \end{aligned} \tag{4.2}$$

where $\langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\mathrm{NA}}$ is the NN t-matrix in the NA c.m. frame.

The NN t-matrix $\langle \boldsymbol{k}', \boldsymbol{p}' | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p} \rangle_{\mathrm{NA}}$ is given by the NN t-matrix in the NN c.m. frame, $\langle \boldsymbol{\kappa}', -\boldsymbol{\kappa}' | t^{(0)}_k | \boldsymbol{\kappa}, -\boldsymbol{\kappa} \rangle_{\mathrm{NN}}$, through the frame transformation as shown in ref. [4]

$$\langle \boldsymbol{k}', \boldsymbol{p}' | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p} \rangle_{\mathrm{NA}} = J(\boldsymbol{k}', \boldsymbol{p}', \boldsymbol{k}, \boldsymbol{p})\, R^{\mathrm{l}}_{\mathrm{sp}}(\boldsymbol{k}', \boldsymbol{p}')\, \langle \boldsymbol{\kappa}', -\boldsymbol{\kappa}' | t^{(0)}_k | \boldsymbol{\kappa}, -\boldsymbol{\kappa} \rangle_{\mathrm{NN}} R^{\mathrm{r}}_{\mathrm{sp}}(\boldsymbol{k}, \boldsymbol{p}) \tag{4.3}$$

where $\boldsymbol{\kappa}$ ($\boldsymbol{\kappa}'$) is the momentum of the incident (exit) nucleon in the NN c.m. frame, and

$$J(\boldsymbol{k}', \boldsymbol{p}', \boldsymbol{k}, \boldsymbol{p}) = \sqrt{\frac{E_{\mathrm{N}}(\kappa) E_{\mathrm{N}_k}(\kappa) E_{\mathrm{N}'}(\kappa') E_{\mathrm{N}'_k}(\kappa')}{E_{\mathrm{N}}(k) E_{\mathrm{N}_k}(p) E_{\mathrm{N}'}(k') E_{\mathrm{N}'_k}(p')}} \tag{4.4}$$

is the Möller factor, and $R^{\mathrm{l}}_{\mathrm{sp}}(\boldsymbol{k}', \boldsymbol{p}')$ and $R^{\mathrm{r}}_{\mathrm{sp}}(\boldsymbol{k}, \boldsymbol{p})$ are the relativistic spin rotation matrices in the incident and exit channels, respectively, and

$$E_\alpha(k) = \sqrt{m^2_\alpha + k^2} \tag{4.5}$$

The momenta $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ are calculated from $\boldsymbol{k}$ and $\boldsymbol{p}$ ($\boldsymbol{k}'$ and $\boldsymbol{p}'$) by the Lorentz transformation. The magnitudes κ and κ' are given by

$$\kappa = \sqrt{\frac{(s_{\mathrm{NN}}(\boldsymbol{k}.\boldsymbol{p}) - m^2_{\mathrm{N}} - m^2_{\mathrm{N}_k})^2 - 4 m^2_{\mathrm{N}} m^2_{\mathrm{N}_k}}{4 s_{\mathrm{NN}}(\boldsymbol{k}, \boldsymbol{p})}} \tag{4.6}$$

$$\kappa' = \sqrt{\frac{(s_{\mathrm{NN}}(\boldsymbol{k}', \boldsymbol{p}') - m^2_{\mathrm{N}'} - m^2_{\mathrm{N}'_k})^2 - 4 m^2_{\mathrm{N}'} m^2_{\mathrm{N}'_k}}{4 s_{\mathrm{NN}}(\boldsymbol{k}', \boldsymbol{p}')}} \tag{4.7}$$

with

$$s_{\mathrm{NN}}(\boldsymbol{p}_1, \boldsymbol{p}_2) = (E_{\mathrm{N}_1}(p_1) + E_{\mathrm{N}_2}(p_2))^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 \tag{4.8}$$

The present version of CRDW neglects the relativistic spin rotation effects, thus sets

$$R^{\mathrm{l}}_{\mathrm{sp}}(\boldsymbol{k}', \boldsymbol{p}') = 1, \qquad R^{\mathrm{r}}_{\mathrm{sp}}(\boldsymbol{k}, \boldsymbol{p}) = 1 \tag{4.9}$$

This is supported by numerical estimations at the incident energy below 500MeV (see ref.[4]).

4.2 NN t-matrix in the NN c.m. frame

There are various expressions for the free NN t-matrix in the NN c.m. frame. In CRDW we adopt the expression similar to KMT[1] [9] as

$$\begin{aligned}
t^{(0)}_{k,\mathrm{NN}}(\boldsymbol{\kappa}',\boldsymbol{\kappa}) &\equiv \langle \boldsymbol{\kappa}', -\boldsymbol{\kappa}' | t^{(0)}_k | \boldsymbol{\kappa}, -\boldsymbol{\kappa}\rangle_{\mathrm{NN}} \\
&= \{A_0 + A_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,\mathbf{1}_0\mathbf{1}_k \\
&+ \{B_0 + B_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}}_{\mathrm{c}}) \\
&+ \{C_0 + C_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,((\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}}_{\mathrm{c}})\mathbf{1}_k + \mathbf{1}_0(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}}_{\mathrm{c}})) \\
&+ \{D_0 + D_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,(\boldsymbol{q}_c\cdot\boldsymbol{Q}_c)\{(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{Q}}_{\mathrm{c}}) + (\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}}_{\mathrm{c}})\} \\
&+ \{E_0 + E_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}}_{\mathrm{c}}) \\
&+ \{F_0 + F_1(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{Q}}_{\mathrm{c}})
\end{aligned} \tag{4.10}$$

with

$$\boldsymbol{q}_{\mathrm{c}} = \boldsymbol{\kappa}' - \boldsymbol{\kappa}, \quad \boldsymbol{Q}_{\mathrm{c}} = \boldsymbol{\kappa}' + \boldsymbol{\kappa}, \quad \hat{\boldsymbol{n}}_{\mathrm{c}} = \frac{\boldsymbol{\kappa}\times\boldsymbol{\kappa}'}{|\ \boldsymbol{\kappa}\times\boldsymbol{\kappa}'\ |} \tag{4.11}$$

The amplitudes $A_i, B_i, C_i, D_i, E_i, F_i$ are the scalar functions of $q_c^2,\ Q_c^2$, and $\boldsymbol{q}_c\cdot\boldsymbol{Q}_c$.

Noting the relations (4.8) and

$$\boldsymbol{q}_c^2 + \boldsymbol{Q}_c^2 = 2(\kappa'^2 + \kappa^2), \qquad \boldsymbol{q}_c\cdot\boldsymbol{Q}_c = \kappa'^2 - \kappa^2 \tag{4.12}$$

and defining the incident kinetic energy in the NN lab. frame, $K^{\mathrm{lab}}_{\mathrm{NN}}$, by the relation

$$s_{\mathrm{NN}}(\boldsymbol{k},\boldsymbol{p}) = (m_{\mathrm{N}} + m_{\mathrm{N}_k})^2 + 2m_{\mathrm{N}_k}K^{\mathrm{lab}}_{\mathrm{NN}} \tag{4.13}$$

we may express the amplitudes by the different set of arguments, $(q_c, Q_c, K^{\mathrm{lab}}_{\mathrm{NN}})$

$$A_i = A_i(q_c, Q_c, K^{\mathrm{lab}}_{\mathrm{NN}}), \quad B_i = B_i(q_c, Q_c, K^{\mathrm{lab}}_{\mathrm{NN}}), \quad \text{etc.} \tag{4.14}$$

On-energy shell From the NN scattering experiments, we can only get the NN t-matrix in the NN c.m. frame on the energy shell, where

$$\kappa = \kappa', \qquad \boldsymbol{q}_c\cdot\boldsymbol{Q}_c = 0, \qquad q_c^2 + Q_c^2 = 4\kappa^2 \tag{4.15}$$

Therefore, the on-energy-shell amplitudes can be expressed only by q_c and $K^{\mathrm{lab}}_{\mathrm{NN}}$ as

$$A_i = A_i(q_c, K^{\mathrm{lab}}_{\mathrm{NN}}), \quad B_i = B_i(q_c, K^{\mathrm{lab}}_{\mathrm{NN}}), \quad \text{etc.} \tag{4.16}$$

and the D_i-terms vanish.

In the free NN scattering measurements, these amplitudes are usually presented as the functions of the scattering angle in the NN c.m. frame, θ_{NN}, and $K^{\mathrm{lab}}_{\mathrm{NN}}$ as

$$A_i = A_i(\theta_{\mathrm{NN}}; K^{\mathrm{lab}}_{\mathrm{NN}}), \quad B_i = B_i(\theta_{\mathrm{NN}}; K^{\mathrm{lab}}_{\mathrm{NN}}), \quad \text{etc.} \tag{4.17}$$

The expressions (4.16) and (4.17) are related by the equation

$$q_c = -2\kappa\sin\left(\frac{\theta_{\mathrm{NN}}}{2}\right) \tag{4.18}$$

[1] KMT used $A, B, C, \cdots$ for the amplitudes of the scattering matrix M, but here we use $A, B, C, \cdots$ for the amplitudes of the t-matrix.

5 Local potential approximation

In eq.(4.2), it is very cumbersome to carry out the integration over $\boldsymbol{k}', \boldsymbol{k}$ and $\boldsymbol{p}_k$, on which the NN t-matrix depends. People often approximate the NN t-matrix as an *energy dependent local potential*, $\mathcal{V}_k(\boldsymbol{r}_0 - \boldsymbol{r}_k; \bar{K}^{\rm lab}_{\rm NN})$. It means that $\langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\rm NA}$ is approximated as the functions only of the momentum transfer

$$\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k} = \boldsymbol{p}_k - \boldsymbol{p}'_k, \tag{5.1}$$

and a fixed incident kinetic energy of the NN system, $\bar{K}^{\rm lab}_{\rm NN}$, namely

$$\langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\rm NA} \longrightarrow t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{q}; \bar{K}^{\rm lab}_{\rm NN}) \longrightarrow \mathcal{V}_k(\boldsymbol{r}_0 - \boldsymbol{r}_k; \bar{K}^{\rm lab}_{\rm NN}) \tag{5.2}$$

For this purpose we introduce following approximations.

5.1 Representative momentum approximation

In the integrations of eq. (4.2), we replace $\boldsymbol{p}_k$ in $\langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\rm NA}$ by a suitably chosen representative momentum $\tilde{\boldsymbol{p}}$, which is assumed to be determined by $\boldsymbol{k}$ and $\boldsymbol{k}'$

$$\boldsymbol{p}_k \longrightarrow \tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}(\boldsymbol{k}, \boldsymbol{k}') \tag{5.3}$$

thus

$$\boldsymbol{p}'_k = \boldsymbol{p}_k - \boldsymbol{q} \longrightarrow \tilde{\boldsymbol{p}}' = \tilde{\boldsymbol{p}} - \boldsymbol{q} = \tilde{\boldsymbol{p}}'(\boldsymbol{k}, \boldsymbol{k}') \tag{5.4}$$

Then the NN t-matrix is approximated by the function only of $\boldsymbol{k}$ and $\boldsymbol{k}'$ as

$$\begin{aligned} \langle \boldsymbol{k}', \boldsymbol{p}'_k | t^{(0)}_k | \boldsymbol{k}, \boldsymbol{p}_k \rangle_{\rm NA} &\approx \langle \boldsymbol{k}', \tilde{\boldsymbol{p}}' | t^{(0)}_k | \boldsymbol{k}, \tilde{\boldsymbol{p}} \rangle_{\rm NA} \\ &= J(\boldsymbol{k}', \tilde{\boldsymbol{p}}', \boldsymbol{k}, \tilde{\boldsymbol{p}})\, t^{(0)}_{k,{\rm NN}}(\tilde{\boldsymbol{\kappa}}', \tilde{\boldsymbol{\kappa}}) \equiv t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{k}', \boldsymbol{k}) \end{aligned} \tag{5.5}$$

where $\tilde{\boldsymbol{\kappa}}$ and $\tilde{\boldsymbol{\kappa}}'$ are determined by $\boldsymbol{k}$ and $\boldsymbol{k}'$.

We call this approximation *representative momentum approximation.*

5.2 Asymptotic momentum approximation

The momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$ distribute around the asymptotic momenta, $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$, respectively, due to distortion, but their distribution should be narrow for high energy small angle scatterings. Therefore, we may use the *asymptotic momentum approximation* (AMA)

$$\boldsymbol{k} \approx \boldsymbol{k}_i, \quad \boldsymbol{k}' \approx \boldsymbol{k}_f \tag{5.6}$$

for quantities whose dependence on $\boldsymbol{k}$ and $\boldsymbol{k}'$ is weak. This is equivalent to use the values, which appears in a plane wave impulse approximation(PWIA).

(1) **Incident kinetic energy of the NN system**

Assuming that the energy dependence of the NN t-matrix is weak, we replace the incident energy $K^{\rm lab}_{\rm NN}$ by a representative energy $\bar{K}^{\rm lab}_{\rm NN}$ obtained by AMA

$$K^{\rm lab}_{\rm NN} \longrightarrow \bar{K}^{\rm lab}_{\rm NN} = \frac{\bar{s}_{\rm NN} - m^2_{\rm N} - m^2_{{\rm N}_k}}{2m_{{\rm N}_k}} \tag{5.7}$$

with

$$\bar{s}_{\rm NN} = s_{\rm NN}(\boldsymbol{k}_i, \bar{\boldsymbol{p}}), \qquad \bar{\boldsymbol{p}} = \tilde{\boldsymbol{p}}(\boldsymbol{k}_i, \boldsymbol{k}_f) \tag{5.8}$$

Note that the quantities with the bar are not operators but the c-numbers.

(2) **Reaction plane**

The reaction plane of the NN scattering fluctuates as $\boldsymbol{k}$ and $\boldsymbol{k}'$ do, but its normal $\hat{\boldsymbol{n}}_c$ should be $\hat{\boldsymbol{n}}$ in average. Therefore we take the replacement

$$\hat{\boldsymbol{n}}_c = \frac{\boldsymbol{\kappa} \times \boldsymbol{\kappa}'}{|\boldsymbol{\kappa} \times \boldsymbol{\kappa}'|} = \frac{\boldsymbol{k} \times \boldsymbol{k}'}{|\boldsymbol{k} \times \boldsymbol{k}'|} \longrightarrow \bar{\hat{\boldsymbol{n}}}_c = \hat{\boldsymbol{n}} = \frac{\boldsymbol{k}_i \times \boldsymbol{k}_f}{|\boldsymbol{k}_i \times \boldsymbol{k}_f|} \tag{5.9}$$

This is nothing but AMA. Again note that $\hat{\boldsymbol{n}}$ is not an operator but the constant vector.

(3) **Möller factor**

We also apply AMA to the Möller factor of eq. (4.4) as

$$J(\boldsymbol{k}', \tilde{\boldsymbol{p}}', \boldsymbol{k}, \tilde{\boldsymbol{p}}) \ \longrightarrow \bar{J} = J(\boldsymbol{k}_f, \bar{\boldsymbol{p}}', \boldsymbol{k}_i, \bar{\boldsymbol{p}}) \tag{5.10}$$

Further using *on-energy shell approximation* ($\kappa' = \kappa$), we get

$$\bar{J} = \frac{E_{\mathrm{N}}(\bar{\kappa}) E_{\mathrm{N}_k}(\bar{\kappa})}{\sqrt{E_{\mathrm{N}}(k_i) E_{\mathrm{N}_k}(\bar{p}) E_{\mathrm{N}'}(k_f) E_{\mathrm{N}'_k}(\bar{p}')}} \tag{5.11}$$

with

$$\bar{\kappa} = \sqrt{\frac{(\bar{s}_{\mathrm{NN}} - m_{\mathrm{N}}^2 - m_{\mathrm{N}_k}^2)^2 - 4 m_{\mathrm{N}_0}^2 m_{\mathrm{N}_k}^2}{4 \bar{s}_{\mathrm{NN}}}} \tag{5.12}$$

5.3 Approximations for the NN t-matrix in the NN c.m. frame

Next we cosider how to localize the NN t-matrix in the NN c.m. system $t^{(0)}_{k,\mathrm{NN}}(\tilde{\boldsymbol{\kappa}}', \tilde{\boldsymbol{\kappa}})$ in eq. (5.5). For this purpose we utilize the following approximations

(1) We discard the D-terms. They are pure off-shell quantities and expected to be small.

(2) For the momentum transfer, we adopt the non-relativistic approximation[2]

$$\boldsymbol{q}_{\mathrm{c}} = \boldsymbol{\kappa}' - \boldsymbol{\kappa} \approx \boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k} \tag{5.13}$$

From these approximations as well as those in sect. 5.1 and 5.2, the NN t-matrix (5.5) is simplified as

$$\begin{aligned} t^{(0)}_{k,\mathrm{NN/A}}(\boldsymbol{k}', \boldsymbol{k}) \ \approx\ & \bar{J}\,[\{A_0 + A_1(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\}\, \mathbf{1}_0 \mathbf{1}_k \\ & + \{(B_0 - F_0) + (B_1 - F_1)(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}) \\ & + \{C_0 + C_1(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\}\,((\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})\mathbf{1}_k + \mathbf{1}_0(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})) \\ & + \{E_0 + E_1(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) \\ & + \{F_0 + F_1(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\}\,(\boldsymbol{\sigma}_0 \times \hat{\boldsymbol{q}}) \cdot (\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}})] \end{aligned} \tag{5.14}$$

with

$$A_i = A_i(q, Q_c, \bar{K}^{\mathrm{lab}}_{\mathrm{NN}}), \quad B_i = B_i(q, Q_c, \bar{K}^{\mathrm{lab}}_{\mathrm{NN}}), \quad \text{etc.} \tag{5.15}$$

[2] As to the direction, this seems to hold very well. Numerical test shows $\hat{\boldsymbol{q}}_{\mathrm{c}} \cdot \hat{\boldsymbol{q}} > 0.996$ for $\theta_{cm} \leq 30$deg and $K_{\mathrm{lab}} \leq 500$ MeV) with $\omega < 100$MeV. As to the magnitudes, the validity depends on the energy transfer, since $t_{\mathrm{NN}} = (E_{\mathrm{N}'}(\kappa') - E_{\mathrm{N}}(\kappa))^2 - q_c^2 = (E_{\mathrm{N}'}(k') - E_{\mathrm{N}}(k))^2 - q^2$.

where we used the relation and the approximation

$$
\begin{aligned}
(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{Q}}_{\rm c})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{Q}}_{\rm c}) &= (\boldsymbol{\sigma}_0 \times \hat{\boldsymbol{q}}_c) \cdot (\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}}_c) - (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_c)(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_c) \\
&\approx (\boldsymbol{\sigma}_0 \times \hat{\boldsymbol{q}}) \cdot (\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}}) - (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}) \qquad (5.16)
\end{aligned}
$$

We emphasize that the spin parts only depend on $\hat{\boldsymbol{q}}$ and the fixed direction $\hat{\boldsymbol{n}}$ in eq. (5.14)

Remaining tasks are to determine the representative momentum $\tilde{\boldsymbol{p}}(\boldsymbol{k}, \boldsymbol{k}')$ and to remove the Q_c-dependence of the amplitudes, A_i, B_i, etc. CRDW provides the two prescriptions:
(1) Optimal factorization prescription [4]
(2) Love-Franey prescription [7]

5.4 Optimal factorization prescription

The one is the optimal factorization prescription, in which the representative momentum $\tilde{\boldsymbol{p}}$ is taken to be

$$
\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}(\boldsymbol{k}, \boldsymbol{k}') = \left(\frac{1}{2} - \eta\right) \boldsymbol{q} - \frac{\boldsymbol{k} + \boldsymbol{k}'}{2A} \qquad (5.17)
$$

The parameter η is determined by the on-energy-shell condition

$$
E_{\rm N_0}(k) + E_{{\rm N}_k}(\tilde{p}) = E_{\rm N_0'}(k') + E_{{\rm N}_k'}(\tilde{p}') \qquad (5.18)
$$

We further apply ANA on η to make it a fixed c-number

$$
\begin{aligned}
\eta \;\approx\; \bar{\eta} &= -\left(\frac{\bar{\boldsymbol{k}}_a \cdot \boldsymbol{q}_{\rm cm}}{A q_{\rm cm}^2} + \frac{m_{\rm N'}^2 - m_{\rm N}^2}{2\bar{t}_{\rm NN}}\right) \\
&+ \frac{\omega_{\rm cm}}{q_{\rm cm}} \sqrt{\frac{1}{4} - \frac{1}{\bar{t}_{\rm NN}}\left(\frac{m_{{\rm N}_k'}^2 + m_{{\rm N}_k}^2}{2} + \frac{\bar{k}_a^2}{A^2} - \left(\frac{\bar{\boldsymbol{k}}_a \cdot \boldsymbol{q}_{\rm cm}}{A q_{\rm cm}}\right)^2 - \frac{1}{\bar{t}_{\rm NN}}\left(\frac{m_{\rm N'}^2 - m_{\rm N}^2}{2}\right)^2\right)} \qquad (5.19)
\end{aligned}
$$

with

$$
\bar{\boldsymbol{k}}_a = \frac{\boldsymbol{k}_i + \boldsymbol{k}_f}{2}, \qquad \bar{t}_{\rm NN} = \omega_{\rm cm}^2 - q_{\rm cm}^2 = t_{\rm NA} \qquad (5.20)
$$

Using $\bar{\eta}$, we calculate $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{p}}'$, and then $\bar{K}_{\rm NN}^{\rm lab}$ and $\bar{J} = J_{\rm opt}$ by (5.7) and (5.11). Due to the on-energy shell condition and the approximation (5.13), we get the amplitudes A_i, B_i, etc. as

$$
A_i = A_i(q; \bar{K}_{\rm NN}^{\rm lab}) \approx A_i(q_c; \bar{K}_{\rm NN}^{\rm lab}) \longleftarrow A_i(\theta_{\rm NN}; \bar{K}_{\rm NN}^{\rm lab}), \quad \text{etc.} \qquad (5.21)
$$

Finally we get

$$
\begin{aligned}
& t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{k}', \boldsymbol{k}) \approx t^{\rm opt}_{k,{\rm NN/A}}(\boldsymbol{q}; \bar{K}_{\rm NN}^{\rm lab}) \\
= \; & J_{\rm opt}\left[\left\{A_0(q; \bar{K}_{\rm NN}^{\rm lab}) + A_1(q; \bar{K}_{\rm NN}^{\rm lab})(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\right\} \mathbf{1}_0 \mathbf{1}_k\right. \\
+ \; & \left\{(B_0(q; \bar{K}_{\rm NN}^{\rm lab}) - F_0(q; \bar{K}_{\rm NN}^{\rm lab})) + (B_1(q; \bar{K}_{\rm NN}^{\rm lab}) - F_1(q; \bar{K}_{\rm NN}^{\rm lab}))(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\right\} (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}) \\
+ \; & \left\{C_0(q; \bar{K}_{\rm NN}^{\rm lab}) + C_1(q; \bar{K}_{\rm NN}^{\rm lab})(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\right\} ((\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})\mathbf{1}_k + \mathbf{1}_0(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})) \\
+ \; & \left\{E_0(q; \bar{K}_{\rm NN}^{\rm lab}) + E_1(q; \bar{K}_{\rm NN}^{\rm lab})(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\right\} (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) \\
+ \; & \left.\left\{F_0(q; \bar{K}_{\rm NN}^{\rm lab}) + F_1(q; \bar{K}_{\rm NN}^{\rm lab})(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\right\} (\boldsymbol{\sigma}_0 \times \hat{\boldsymbol{q}}) \cdot (\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}})\right] \qquad (5.22)
\end{aligned}
$$

which only depends on $\boldsymbol{q}$ and $\bar{K}_{\rm NN}^{\rm lab}$.

We must note that the on-energy shell condition requires $\theta_{\rm NN} \leq \pi$, thus

$$
q \leq 2\overline{\kappa} = \sqrt{2 m_{\rm N} \bar{K}_{\rm NN}^{\rm lab}} \qquad (5.23)
$$

due to eq.(4.18). This could be violated at large angle scatterings.

5.5 Love-Franey prescription

The other is the Love-Franey's prescription [7], [8].

5.5.1 Free NN t-matrix

They proposed a phenomenological free NN t-matrix, which works for the off-energy shell, and is made to reproduce the on-energy shell data. It has the form of

$$
\begin{aligned}
t^{(0)}_{k,\mathrm{NN}}(\boldsymbol{\kappa}',\boldsymbol{\kappa}) &\approx t^{\mathrm{LF}}_{k,\mathrm{NN}}(\boldsymbol{\kappa}',\boldsymbol{\kappa}) \\
&= \left[\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_{\mathrm{c}}) - \tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_{\mathrm{c}})\right] P_{S=0}P_{T=0} + \left[\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q_{\mathrm{c}}) + \tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(Q_{\mathrm{c}})\right] P_{S=0}P_{T=1} \\
&+ \left[\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_{\mathrm{c}}) - \tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_{\mathrm{c}})\right] P_{S=1}P_{T=1} + \left[\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_{\mathrm{c}}) + \tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_{\mathrm{c}})\right] P_{S=1}P_{T=0} \\
&+ \frac{\mathrm{i}}{4}\left[Q_{\mathrm{c}}\tilde{V}^{\mathrm{LSO}}(q_{\mathrm{c}}) + q_{\mathrm{c}}\tilde{V}^{\mathrm{LSO}}(Q_{\mathrm{c}})\right] \left((\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_k)\cdot\hat{\boldsymbol{n}}_{\mathrm{c}}\right) P_{T=1} \\
&+ \frac{\mathrm{i}}{4}\left[Q_{\mathrm{c}}\tilde{V}^{\mathrm{LSE}}(q_{\mathrm{c}}) - q_{\mathrm{c}}\tilde{V}^{\mathrm{LSE}}(Q_{\mathrm{c}})\right] \left((\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_k)\cdot\hat{\boldsymbol{n}}_{\mathrm{c}}\right) P_{T=0} \\
&- \left[\tilde{V}^{\mathrm{TNO}}(q_{\mathrm{c}})S_{0k}(\hat{\boldsymbol{q}}_{\mathrm{c}}) - \tilde{V}^{\mathrm{TNO}}(Q_{\mathrm{c}})S_{0k}(\hat{\boldsymbol{Q}}_{\mathrm{c}})\right] P_{T=1} \\
&- \left[\tilde{V}^{\mathrm{TNE}}(q_{\mathrm{c}})S_{0k}(\hat{\boldsymbol{q}}_{\mathrm{c}}) + \tilde{V}^{\mathrm{TNE}}(Q_{\mathrm{c}})S_{0k}(\hat{\boldsymbol{Q}}_{\mathrm{c}})\right] P_{T=0}
\end{aligned} \tag{5.24}
$$

with

$$
\tilde{V}^{\mathrm{C}}_{S/T\,\Pi}(\lambda) = 4\pi \sum_i^{N_C} \frac{V^{\mathrm{C}}_{S/T\,\Pi,i}(R^{\mathrm{C}}_i)^3}{1+(\lambda R^{\mathrm{C}}_i)^2} \tag{5.25}
$$

$$
\tilde{V}^{\mathrm{LS}\Pi}(\lambda) = 8\pi \sum_i^{N_{LS}} \frac{V^{\mathrm{LS},\Pi}_i \lambda (R^{\mathrm{LS}}_i)^5}{[1+(\lambda R^{\mathrm{LS}}_i)^2]^2}, \qquad \tilde{V}^{\mathrm{TN}\Pi}(\lambda) = 32\pi \sum_i^{N_{TN}} \frac{V^{\mathrm{TN},\Pi}_i \lambda^2 (R^{\mathrm{TN}}_i)^7}{[1+(\lambda R^{\mathrm{TN}}_i)^2]^3} \tag{5.26}
$$

where $\Pi = \mathrm{O(DD)}$ or $\mathrm{E(VEN)}$, $\lambda = q_c$ or Q_c, and R^α_i ($\alpha = \mathrm{C}, \mathrm{LS}, \mathrm{TN}$) are the range parameters, and N^α ($\alpha = \mathrm{C}, \mathrm{LS}, \mathrm{TN}$) are the number of terms with different ranges, and the projection operators are defined as

$$
P_{S=0} = \frac{1-\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_k}{4}, \qquad P_{S=1} = \frac{3+\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_k}{4} \tag{5.27}
$$

and similarly for $P_{T=0}$ and $P_{T=1}$. The tensor operator is defined as

$$
S_{0k}(\hat{\boldsymbol{q}}) = 3(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}}) - (\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_k) \tag{5.28}
$$

Rewriting eq. (5.24) into the form of eq. (4.10), we get the amplitudes [5] as

$$
\begin{aligned}
A_0 &= \frac{1}{16}\left[\left(\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_c) - \tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_c)\right) + 3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q_c) + \tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(Q_c)\right)\right. \\
&+ \left. 9\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_c) - \tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_c)\right) + 3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_c) + \tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_c)\right)\right]
\end{aligned} \tag{5.29}
$$

$$
\begin{aligned}
A_1 &= \frac{1}{16}\left[-\left(\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_c) - \tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_c)\right) + \left(\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q_c) + \tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(Q_c)\right)\right. \\
&+ \left. 3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_c) - \tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_c)\right) - 3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_c) + \tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_c)\right)\right]
\end{aligned} \tag{5.30}
$$

$$\begin{aligned}
B_0 &= \frac{1}{16}\Big[-\left(\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_c)-\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_c)\right)-3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q_c)+\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(Q_c)\right)\\
&+ 3\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_c)-\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_c)\right)+\left(\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_c)+\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_c)\right)\\
&+ 4\left(3\tilde{V}^{\mathrm{TNO}}(q_c)+\tilde{V}^{\mathrm{TNE}}(q_c)-3\tilde{V}^{\mathrm{TNO}}(Q_c)+\tilde{V}^{\mathrm{TNE}}(Q_c)\right)\Big] \qquad (5.31)\\
B_1 &= \frac{1}{16}\Big[\Big[\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_c)-\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_c)\Big]-\Big[\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q)+\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(k_{\mathrm{i}})\Big]\\
&+ \Big[\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_c)-\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_c)\Big]-\Big[\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_c)+\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_c)\Big]\\
&+ 4\left(\tilde{V}^{\mathrm{TNO}}(q_c)-\tilde{V}^{\mathrm{TNE}}(q_c)-\tilde{V}^{\mathrm{TNO}}(Q_c)-\tilde{V}^{\mathrm{TNE}}(Q_c)\right)\Big] \qquad (5.32)\\
C_0 &= \frac{\mathrm{i}}{16}\Big[3\left(Q_c\tilde{V}^{\mathrm{LSO}}(q_c)+q_c\tilde{V}^{\mathrm{LSO}}(Q_c)\right)+\left(Q_c\tilde{V}^{\mathrm{LSE}}(q_c)-q_c\tilde{V}^{\mathrm{LSE}}(Q_c)\right)\Big] \qquad (5.33)\\
C_1 &= \frac{\mathrm{i}}{16}\Big[\left(Q_c\tilde{V}^{\mathrm{LSO}}(q_c)+q_c\tilde{V}^{\mathrm{LSO}}(Q_c)\right)-\left(Q_c\tilde{V}^{\mathrm{LSE}}(q_c)-q_c\tilde{V}^{\mathrm{LSE}}(Q_c)\right)\Big] \qquad (5.34)\\
E_0 &= B_0-\frac{3}{4}\Big[3\tilde{V}^{\mathrm{TNO}}(q_c)+\tilde{V}^{\mathrm{TNE}}(q_c)\Big] \qquad (5.35)\\
E_1 &= B_1-\frac{3}{4}\Big[\tilde{V}^{\mathrm{TNO}}(q_c)-\tilde{V}^{\mathrm{TNE}}(q_c)\Big] \qquad (5.36)\\
F_0 &= B_0+\frac{3}{4}\Big[3\tilde{V}^{\mathrm{TNO}}(Q_c)-\tilde{V}^{\mathrm{TNE}}(Q_c)\Big] \qquad (5.37)\\
F_1 &= B_1+\frac{3}{4}\Big[\tilde{V}^{\mathrm{TNO}}(Q_c)+\tilde{V}^{\mathrm{TNE}}(Q_c)\Big] \qquad (5.38)
\end{aligned}$$

5.5.2 NN t-matrix in the NA frame

This method takes the representative momentum as

$$\bar{\boldsymbol{p}} = \tilde{\boldsymbol{p}} = -\frac{\boldsymbol{k}_i}{A} \qquad (5.39)$$

which means neglect of the Fermi motion. Neglecting the nucleon mass difference, we get $\bar{K}^{\mathrm{lab}}_{\mathrm{NN}}$ by eq. (5.7) with

$$\bar{s}_{\mathrm{NN}} = (E_{\mathrm{N}}(k_i)+E_{\mathrm{N}}(k_i/A))^2-(k_i-k_i/A)^2 \qquad (5.40)$$

The Möller factor is fixed to the value for the elastic forward scattering ($\boldsymbol{q}=0,\ \omega_{\mathrm{cm}}=0,\ \bar{\boldsymbol{p}}=\bar{\boldsymbol{p}}'$)) as[3]

$$\bar{J} = J_{\mathrm{LF}} = \frac{E^2_{\mathrm{N}}(\bar{\kappa})}{E_{\mathrm{N}}(k_i)E_{\mathrm{N}}(k_i/A)} \qquad (5.41)$$

by use of eqs. (5.11) and (5.12)

To remove Q_c dependence from the amplitudes, this method uses the *pseudo-potential* approximation for the exchange terms, in which they are represented by the δ-type potentials in the coordinate space. This means that Q_c is replaced by a certain constant, which is chosen to be the value of the elastic forward scattering ($\boldsymbol{q}=0, \omega_{\mathrm{c}}=0$) in AMA

$$Q_c = |\boldsymbol{\kappa}+\boldsymbol{\kappa}'| \longrightarrow \bar{Q}_c = 2\bar{\kappa} \qquad (5.42)$$

Further they take the infinite target mass limit ($m_A\to\infty$) and the non-relativistic approximations, then get

$$Q_c \longrightarrow \bar{Q}_c = k_{\mathrm{i}} \qquad (5.43)$$

[3]In refs. [7] and [5], the notations $\epsilon_{\mathrm{cm}}=E_{\mathrm{N}}(\bar{\kappa}),\ \epsilon_{\mathrm{i}}=E_{\mathrm{N}}(k_i),\ \epsilon_{\mathrm{t}}=E_{\mathrm{N}}(k_i/A)$ are used.

From these approximations, the amplitudes $A_i, \cdots$ are given by the functions only of q as

$$A_i = A_i(q_c, Q_c; K_{\rm NN}^{\rm lab}) \approx A_i(q, k_i; \bar{K}_{\rm NN}^{\rm lab}) = A_i(q; \bar{K}_{\rm NN}^{\rm lab}), \quad \text{etc.} \tag{5.44}$$

At the end, the NN t matrix in the NA c.m. system (5.14) is given in the form of

$$\begin{aligned}
& t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{k}', \boldsymbol{k}) \approx t^{\rm LF}_{k,{\rm NN/A}}(\boldsymbol{q}; \bar{K}_{\rm NN}^{\rm lab}) \ = J_{\rm LF} t^{\rm LF}_{k,{\rm NN}}(\boldsymbol{q}; \bar{K}_{\rm NN}^{\rm lab}) \\
= \;& J_{\rm LF}\left[\left\{A_0(q; \bar{K}_{\rm NN}^{\rm lab}) + A_1(q; \bar{K}_{\rm NN}^{\rm lab})\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k\right\} \mathbf{1}_0 \mathbf{1}_k\right. \\
+ \;& \left\{(B_0(q; \bar{K}_{\rm NN}^{\rm lab}) - F_0(q; \bar{K}_{\rm NN}^{\rm lab})) + (B_1(q; \bar{K}_{\rm NN}^{\rm lab}) - F_1(q; \bar{K}_{\rm NN}^{\rm lab}))\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k\right\} (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}) \\
+ \;& \left\{C_0(q; \bar{K}_{\rm NN}^{\rm lab}) + C_1(q; \bar{K}_{\rm NN}^{\rm lab})\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k\right\} ((\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})\mathbf{1}_k + \mathbf{1}_0(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})) \\
+ \;& \left\{E_0(q; \bar{K}_{\rm NN}^{\rm lab}) + E_1(q; \bar{K}_{\rm NN}^{\rm lab})\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k\right\} (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) \\
+ \;& \left.\left\{F_0(q; \bar{K}_{\rm NN}^{\rm lab}) + F_1(q; \bar{K}_{\rm NN}^{\rm lab})\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k\right\} (\boldsymbol{\sigma}_0 \times \hat{\boldsymbol{q}}) \cdot (\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}})\right]
\end{aligned} \tag{5.45}$$

with

$$\begin{aligned}
A_0(q; \bar{K}_{\rm NN}^{\rm lab}) &= \frac{1}{16}\left[\left(\tilde{V}^{\rm C}_{\rm SO}(q) - \tilde{V}^{\rm C}_{\rm SO}(k_{\rm i})\right) + 3\left(\tilde{V}^{\rm C}_{\rm SE}(q) + \tilde{V}^{\rm C}_{\rm SE}(k_{\rm i})\right)\right. \\
&+ \left. 9\left(\tilde{V}^{\rm C}_{\rm TO}(q) - \tilde{V}^{\rm C}_{\rm TO}(k_{\rm i})\right) + 3\left(\tilde{V}^{\rm C}_{\rm TE}(q) + \tilde{V}^{\rm C}_{\rm TE}(k_{\rm i})\right)\right]
\end{aligned} \tag{5.46}$$

$$\begin{aligned}
A_1(q; \bar{K}_{\rm NN}^{\rm lab}) &= \frac{1}{16}\left[-\left(\tilde{V}^{\rm C}_{\rm SO}(q) - \tilde{V}^{\rm C}_{\rm SO}(k_{\rm i})\right) + \left(\tilde{V}^{\rm C}_{\rm SE}(q) + \tilde{V}^{\rm C}_{\rm SE}(k_{\rm i})\right)\right. \\
&+ \left. 3\left(\tilde{V}^{\rm C}_{\rm TO}(q) - \tilde{V}^{\rm C}_{\rm TO}(k_{\rm i})\right) - 3\left(\tilde{V}^{\rm C}_{\rm TE}(q) + \tilde{V}^{\rm C}_{\rm TE}(k_{\rm i})\right)\right]
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
B_0(q; \bar{K}_{\rm NN}^{\rm lab}) &= \frac{1}{16}\left[-\left(\tilde{V}^{\rm C}_{\rm SO}(q) - \tilde{V}^{\rm C}_{\rm SO}(k_{\rm i})\right) - 3\left(\tilde{V}^{\rm C}_{\rm SE}(q) + \tilde{V}^{\rm C}_{\rm SE}(k_{\rm i})\right)\right. \\
&+ 3\left(\tilde{V}^{\rm C}_{\rm TO}(q) - \tilde{V}^{\rm C}_{\rm TO}(k_{\rm i})\right) + \left(\tilde{V}^{\rm C}_{\rm TE}(q) + \tilde{V}^{\rm C}_{\rm TE}(k_{\rm i})\right) \\
&+ \left. 4\left(3\tilde{V}^{\rm TNO}(q) + \tilde{V}^{\rm TNE}(q) - 3\tilde{V}^{\rm TNO}(k_{\rm i}) + \tilde{V}^{\rm TNE}(k_{\rm i})\right)\right]
\end{aligned} \tag{5.48}$$

$$\begin{aligned}
B_1(q; \bar{K}_{\rm NN}^{\rm lab}) &= \frac{1}{16}\left[\left[\tilde{V}^{\rm C}_{\rm SO}(q) - \tilde{V}^{\rm C}_{\rm SO}(k_{\rm i})\right] - \left[\tilde{V}^{\rm C}_{\rm SE}(q) + \tilde{V}^{\rm C}_{\rm SE}(k_{\rm i})\right]\right. \\
&+ \left[\tilde{V}^{\rm C}_{\rm TO}(q) - \tilde{V}^{\rm C}_{\rm TO}(k_{\rm i})\right] - \left[\tilde{V}^{\rm C}_{\rm TE}(q) + \tilde{V}^{\rm C}_{\rm TE}(k_{\rm i})\right] \\
&+ \left. 4\left(\tilde{V}^{\rm TNO}(q) - \tilde{V}^{\rm TNE}(q) - \tilde{V}^{\rm TNO}(k_{\rm i}) - \tilde{V}^{\rm TNE}(k_{\rm i})\right)\right]
\end{aligned} \tag{5.49}$$

$$C_0(q; \bar{K}_{\rm NN}^{\rm lab}) = \frac{\rm i}{16}\left[3\left(k_{\rm i}\tilde{V}^{\rm LSO}(q) + q\tilde{V}^{\rm LSO}(k_{\rm i})\right) + \left(k_{\rm i}\tilde{V}^{\rm LSE}(q) - q\tilde{V}^{\rm LSE}(k_{\rm i})\right)\right] \tag{5.50}$$

$$C_1(q; \bar{K}_{\rm NN}^{\rm lab}) = \frac{\rm i}{16}\left[\left(k_{\rm i}\tilde{V}^{\rm LSO}(q) + q\tilde{V}^{\rm LSO}(k_{\rm i})\right) - \left(k_{\rm i}\tilde{V}^{\rm LSE}(q) - q\tilde{V}^{\rm LSE}(k_{\rm i})\right)\right] \tag{5.51}$$

$$E_0(q; \bar{K}_{\rm NN}^{\rm lab}) = B_0(q; \bar{K}_{\rm NN}^{\rm lab}) - \frac{3}{4}\left[3\tilde{V}^{\rm TNO}(q) + \tilde{V}^{\rm TNE}(q)\right] \tag{5.52}$$

$$E_1(q; \bar{K}_{\rm NN}^{\rm lab}) = B_1(q; \bar{K}_{\rm lab}) - \frac{3}{4}\left[\tilde{V}^{\rm TNO}(q) - \tilde{V}^{\rm TNE}(q)\right] \tag{5.53}$$

$$F_0(q; \bar{K}_{\rm NN}^{\rm lab}) = B_0(q; \bar{K}_{\rm lab}) + \frac{3}{4}\left[3\tilde{V}^{\rm TNO}(k_{\rm i}) - \tilde{V}^{\rm TNE}(k_{\rm i})\right] \tag{5.54}$$

$$F_1(q; \bar{K}_{\rm NN}^{\rm lab}) = B_1(q; \bar{K}_{\rm NN}^{\rm lab}) + \frac{3}{4}\left[\tilde{V}^{\rm TNO}(k_{\rm i}) + \tilde{V}^{\rm TNE}(k_{\rm i})\right] \tag{5.55}$$

The program uses the parameters given in [8]. It has an option that treats $\bar{K}_{\rm NN}^{\rm lab}$ as an input parameter free from the calculated one.

5.6 Driving Force – Energy dependent local potential

As was given in eq. (5.22) or (5.45), we now obtained the NN t-matrix in the NA system, which depends only on the momentum transfer $\boldsymbol{q}$ for the given energy $\bar{K}^{\rm lab}_{\rm NN}$ as

$$\begin{aligned}
& t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{k}',\boldsymbol{k}) \approx t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{q};\bar{K}^{\rm lab}_{\rm NN}) \\
= \ & \bar{J}\left[\left\{A_0(q;\bar{K}^{\rm lab}_{\rm NN}) + A_1(q;\bar{K}^{\rm lab}_{\rm NN})(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\right\}\mathbf{1}_0\mathbf{1}_k\right. \\
+ \ & \left\{(B_0(q;\bar{K}^{\rm lab}_{\rm NN}) - F_0(q;\bar{K}^{\rm lab}_{\rm NN})) + (B_1(q;\bar{K}^{\rm lab}_{\rm NN}) - F_1(q;\bar{K}^{\rm lab}_{\rm NN}))(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\right\}(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}}) \\
+ \ & \left\{C_0(q;\bar{K}^{\rm lab}_{\rm NN}) + C_1(q;\bar{K}^{\rm lab}_{\rm NN})(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\right\}((\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}})\mathbf{1}_k + \mathbf{1}_0(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}})) \\
+ \ & \left\{E_0(q;\bar{K}^{\rm lab}_{\rm NN}) + E_1(q;\bar{K}^{\rm lab}_{\rm NN})(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\right\}(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}}) \\
+ \ & \left.\left\{F_0(q;\bar{K}^{\rm lab}_{\rm NN}) + F_1(q;\bar{K}^{\rm lab}_{\rm NN})(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_k)\right\}(\boldsymbol{\sigma}_0\times\hat{\boldsymbol{q}})\cdot(\boldsymbol{\sigma}_k\times\hat{\boldsymbol{q}})\right]
\end{aligned} \tag{5.56}$$

Moving to the coordinate space, we finally obtain the energy dependent local potential

$$\mathcal{V}_k(\boldsymbol{r}_0-\boldsymbol{r}_k;\bar{K}_{\rm lab}) = \int\frac{d\boldsymbol{q}^3}{(2\pi)^3}{\rm e}^{{\rm i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)}t^{(0)}_{k,{\rm NN/A}}(\boldsymbol{q};\bar{K}^{\rm lab}_{\rm NN}) \tag{5.57}$$

which we call a *driving force* potential because it initiates the reaction.

6 Angular momentum representation

In the practical calculation, we use the $[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ coordinate system and the angular momentum representation, for which we introduce the spherical tensor representation of the nucleon spin and isospin operators as

$$\sigma^{(0)}_0 = \mathbf{1},\quad \sigma^{(1)}_0 = \sigma_z,\quad \sigma^{(1)}_{\pm1} = \mp\frac{\sigma_x \pm {\rm i}\sigma_y}{\sqrt{2}} \tag{6.1}$$

$$\tau^{(0)}_0 = \mathbf{1},\quad \tau^{(1)}_0 = \tau_z,\quad \tau^{(1)}_{\pm1} = \mp\frac{\tau_x \pm {\rm i}\tau_y}{\sqrt{2}} \tag{6.2}$$

and that of the reaction plane normal as

$$\hat{n}_0 = \hat{n}_z = 0,\quad \hat{n}_{\pm1} = \mp\frac{\hat{n}_x \pm {\rm i}\hat{n}_y}{\sqrt{2}} = -\frac{\rm i}{\sqrt{2}} \tag{6.3}$$

6.1 NA T-matrix

The driving force potential (5.57) can generally be expanded by the spherical tensors as[4]

$$\begin{aligned}
\mathcal{V}_k(\boldsymbol{r}_0-\boldsymbol{r}_k) \ = \ & \sum_{t\nu}\sum_{l's'J'M'}\sum_{lsJM} V^{(t)}(r_0,r_k;l's'J'M';lsJM) \\
\times \ & \left(\tau^{(t)}_{\nu,0}\left[{\rm i}^{l'}Y_{l'}(\hat{\boldsymbol{r}}_0)\times\sigma^{(s')}_0\right]^{J'}_{M'}\right)\left(\tau^{(t)}_{\nu,k}\left[{\rm i}^{l}Y_{l}(\hat{\boldsymbol{r}}_k)\times\sigma^{(s)}_k\right]^{J}_{M}\right)^{\dagger}
\end{aligned} \tag{6.5}$$

Here and later we suppress the argument $\bar{K}^{\rm lab}_{\rm NN}$ for simplicity.

[4]The tensor product is defined as

$$\left[A^{J_1}\times B^{J_2}\right]^J_M = \sum_{M_1M_2}(J_1M_1J_2M_2|JM)A^{J_1}_{M_1}B^{J_2}_{M_2} \tag{6.4}$$

Using eq. (6.5), we can write the DWIA T-matrix (4.1) as

$$
\begin{aligned}
T_{n0}^{fi} &= \langle \chi_f \Psi_{\mathrm{X}}^n | \sum_k \mathcal{V}_k(\boldsymbol{r}_0 - \boldsymbol{r}_k) | \Phi_{\mathrm{A}}^0 \chi_i \rangle \\
&= \sum_{t\nu} \sum_{lsJM} \int r^2 dr \; S_{lsJM,t\nu}^{fi}(r) \langle \Psi_{\mathrm{X}}^n | \rho_{lsJM,t\nu}^{\dagger}(r) | \Psi_{\mathrm{A}}^0 \rangle
\end{aligned} \tag{6.6}
$$

where

$$
\begin{aligned}
&S_{lsJM,t\nu}^{fi}(r) \\
&\equiv \sum_{l's'J'M'} \langle \chi_f(\boldsymbol{r}_0) | V^{(t)}(r_0, r; l's'J'M'; lsJM) \left(\tau_{\nu,0}^{(t)} \left[\mathrm{i}^{l'} Y_{l'}(\hat{\boldsymbol{r}}_0) \times \sigma_0^{(s')} \right]_{M'}^{J'} \right) | \chi_i(\boldsymbol{r}_0) \rangle
\end{aligned} \tag{6.7}
$$

is the *outer impulse function* in the angular momentum representation, and

$$
\rho_{lsJM,t\nu}(r) \equiv \sum_k \left(\tau_{\nu,k}^{(t)} \left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_k) \times \sigma_k^{(s)} \right]_M^J \right) \frac{\delta(r - r_k)}{r r_k} \tag{6.8}
$$

is the radial part of the spin-isospin density operator. We note the relation

$$
\rho_{\nu\mu}^{ts}(\boldsymbol{r}) \equiv \sum_k \tau_{\nu,k}^{(t)} \sigma_{\mu,k}^{(s)} \delta(\boldsymbol{r} - \boldsymbol{r}_k) = \sum_{JM} \sum_{lm} (lms\mu | JM) \left(\mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}}) \right)^* \rho_{lsJM,t\nu}(r) \tag{6.9}
$$

6.2 Driving force potential

According to eq. (5.56), we decompose the driving force (5.57) into the terms comming from the $A_t, E_t, F_t, (B_t - F_t)$ and C_t amplitudes, respectively, as

$$
\begin{aligned}
\mathcal{V}_k(\boldsymbol{r}_0 - \boldsymbol{r}_k) &= \sum_t \tau_{\nu,0}^{(t)} \left(\tau_{\nu,k}^{(t)} \right)^{\dagger} \Big[\mathcal{V}_k^{(t),A}(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \mathcal{V}_k^{(t),E}(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \mathcal{V}_k^{(t),F}(\boldsymbol{r}_0 - \boldsymbol{r}_k) \\
&+ \mathcal{V}_k^{(t),B-F}(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \mathcal{V}_k^{(t),C1}(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \mathcal{V}_k^{(t),C2}(\boldsymbol{r}_0 - \boldsymbol{r}_k) \Big]
\end{aligned} \tag{6.10}
$$

each term of which can commonly be expanded as

$$
\begin{aligned}
\mathcal{V}_k^{(t),X}(\boldsymbol{r}_0 - \boldsymbol{r}_k) &= \sum_{l's'J'M'} \sum_{lsJM} V_X^{(t)}(r_0, r_k; l's'J'M'; lsJM) \\
&\times \left(\left[\mathrm{i}^{l'} Y_{l'}(\hat{\boldsymbol{r}}_0) \times \sigma_0^{(s')} \right]_{M'}^{J'} \right) \left(\left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_k) \times \sigma_k^{(s)} \right]_M^J \right)^{\dagger}
\end{aligned} \tag{6.11}
$$

where X denotes $A, E, F, B-F, C1$ or $C2$.

Finally the full radial part $V^{(t)}(r_0, r_k; l's'J'M'; lsJM)$ of eq. (6.5) is given by

$$
V^{(t)}(r_0, r_k; l's'J'M'; lsJM) = \sum_X V_X^{(t)}(r_0, r_k; l's'J'M'; lsJM) \tag{6.12}
$$

Each strength $V_X^{(t)}(r_0, r_k; l's'J'M'; lsJM)$ is given by the following formula.

(1) ***A*-term** The potential coming from the A_t amplitude is written as

$$
\begin{aligned}
\mathcal{V}_k^{(t),A}(\boldsymbol{r}_0 - \boldsymbol{r}_k) &= \bar{J} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} A_t(q) \mathbf{1}_0 \mathbf{1}_k \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0 - \boldsymbol{r}_k)} \\
&= \bar{J} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} A_t(q) \left(\sigma_{0,0}^{(0)} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_0} \right) \left(\sigma_{0,k}^{(0)} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} \right)^{\dagger} \\
&= \sum_J V_A^{(t)}(r_0, r_k; J) \sum_M \left(\left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_0) \times \sigma_{0,0}^{(0)} \right]_M^J \right) \left(\left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_k) \times \sigma_{0,k}^{(0)} \right]_M^J \right)^{\dagger}
\end{aligned} \tag{6.13}
$$

where

$$V_A^{(t)}(r_0,r_k;J) = \bar{J}\frac{2}{\pi}\int_0^\infty A_t(q)j_J(qr_0)j_J(qr_k)q^2dq. \tag{6.14}$$

Here we used the formula

$$\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} = 4\pi\sum_l \mathrm{i}^l j_l(qr)\sum_m Y_{lm}^*(\hat{\boldsymbol{q}})Y_{lm}(\hat{\boldsymbol{r}}) \tag{6.15}$$

Thus we get

$$V_A^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{J'J}\delta_{l'l}\delta_{s's}\delta_{M'M}\delta_{s0}\delta_{Jl}\,V_A^{(t)}(r_0,r_k;J) \tag{6.16}$$

(2) E**-term** The potential coming from the E_t amplitude is written as

$$\begin{aligned}
\mathcal{V}_k^{(t),E}(\boldsymbol{r}_0-\boldsymbol{r}_k) &= \bar{J}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}E_t(q)(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)} \\
&= \sum_{l'lJ}V_E^{(t)}(r_0,r_k;l'lJ)\sum_M\left(\left[\mathrm{i}^{l'}Y_{l'}(\hat{\boldsymbol{r}}_0)\times\sigma_0^{(1)}\right]_M^J\right)\left(\left[\mathrm{i}^{l}Y_{l}(\hat{\boldsymbol{r}}_k)\times\sigma_k^{(1)}\right]_M^J\right)^\dagger
\end{aligned} \tag{6.17}$$

where

$$V_E^{(t)}(r_0,r_k;l'lJ) = a_{Jl'}a_{Jl}\,\bar{J}\frac{2}{\pi}\int_0^\infty E_t(q)j_{l'}(qr_0)j_l(qr_k)q^2dq \tag{6.18}$$

with

$$a_{Jl} = (J010|l0) \tag{6.19}$$

Here we used the formula

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} = -4\pi\sum_{Jl}a_{Jl}\,j_l(qr)\sum_M Y_{JM}^*(\hat{\boldsymbol{q}})\left[\mathrm{i}^lY_l(\hat{\boldsymbol{r}})\times\sigma^{(1)}\right]_M^J \tag{6.20}$$

Thus we get

$$V_E^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{J'J}\delta_{s's}\delta_{M'M}\delta_{s1}\,V_E^{(t)}(r_0,r_k;l'lJ) \tag{6.21}$$

with $l=J\pm1,\ l'=J\pm1$.

(3) F**-term** The potential coming from the F_t amplitude is written as

$$\begin{aligned}
\mathcal{V}_k^{(t)),F}(\boldsymbol{r}_0-\boldsymbol{r}_k) &= \bar{J}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}F(q)(\boldsymbol{\sigma}_0\times\hat{\boldsymbol{q}})\cdot(\boldsymbol{\sigma}_k\times\hat{\boldsymbol{q}})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)} \\
&= \bar{J}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}F(q)\left(\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_k-(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{q}})\right)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)} \\
&= \sum_{l'lJ}V_F^{(t)}(r_0,r_k;l'lJ)\sum_M\left(\left[\mathrm{i}^{l'}Y_{l'}(\hat{\boldsymbol{r}}_0)\times\sigma_0^{(1)}\right]_M^J\right)\left(\left[\mathrm{i}^{l}Y_{l}(\hat{\boldsymbol{r}}_k)\times\sigma_k^{(1)}\right]_M^J\right)^\dagger
\end{aligned} \tag{6.22}$$

where

$$V_F^{(t)}(r_0,r_k;l'lJ) = (\delta_{l'l}-a_{Jl'}a_{Jl})\bar{J}\frac{2}{\pi}\int_0^\infty F_t(q)j_{l'}(qr_0)j_l(qr_k)q^2dq \tag{6.23}$$

Thus we get

$$V_F^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{J'J}\delta_{s's}\delta_{M'M}\delta_{s1}\,V_F^{(t)}(r_0,r_k;l'lJ) \tag{6.24}$$

with $(-1)^l = (-1)^{l'}$.

(4) $(B-F)$**-term** The potential coming from the $B_t - F_t$ amplitude is written as

$$\mathcal{V}_k^{(t),B-F}(\boldsymbol{r}_0-\boldsymbol{r}_k) = \bar{J}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\left(B_t(q)-F_t(q)\right)(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)}$$
$$= \sum_{JM}\sum_{J'M'}\sum_{l} V_{B-F}^{(t)}(r_0,r_k;lJ'M'JM)\left(\left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_0)\times\sigma_0^{(1)}\right]^{J'}_{M'}\right)\left(\left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_k)\times\sigma_k^{(1)}\right]^{J}_{M}\right)^{\dagger} \tag{6.25}$$

where

$$V_{B-F}^{(t)}(r_0,r_k;lJ'M'JM) = \left(\sum_{m\mu'\mu}(lm\ 1\mu'|J'M')(lm\ 1\mu|JM)\hat{n}^{\dagger}_{\mu'}\hat{n}_{\mu}\right) \times \bar{J}\frac{2}{\pi}\int_0^{\infty}\left(B_t(q)-F_t(q)\right)j_l(qr_0)j_l(qr_k)q^2dq \tag{6.26}$$

Thus we get

$$V_{B-F}^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{l'l}\delta_{s's}\delta_{s1}\,V_{B-F}^{(t)}(r_0,r_k;lJ'M'JM) \tag{6.27}$$

with $J,J' = l\pm 1, l$.

(4) $C1$**-term** The first potential coming from the C_t amplitude is written as

$$\mathcal{V}_k^{(t),C1}(\boldsymbol{r}_0-\boldsymbol{r}_k) = \bar{J}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}C_t(q)(\boldsymbol{\sigma}_0\cdot\hat{\boldsymbol{n}})\boldsymbol{1}_k\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)}$$
$$= \sum_{lm}\sum_{J'M'} V_{C1}^{(t)}(r_0,r_k;J'M'JM)\left(\left[\mathrm{i}^J Y_J(\hat{\boldsymbol{r}}_0)\times\sigma_0^{(1)}\right]^{J'}_{M'}\right)\left(\left[\mathrm{i}^J Y_J(\hat{\boldsymbol{r}}_k)\times\sigma_k^{(0)}\right]^{J}_{M}\right)^{\dagger} \tag{6.28}$$

where

$$V_{C1}^{(t)}(r_0,r_k;J'M'JM) = \sum_{\mu}(JM\ 1\ \mu|J'M')\hat{n}^{\dagger}_{\mu}\,\bar{J}\frac{2}{\pi}\int_0^{\infty}C_t(q)j_J(qr_0)j_J(qr_k)q^2dq \tag{6.29}$$

Thus we get

$$V_{C1}^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{Jl}\delta_{l'l}\delta_{s'1}\delta_{s0}\,V_{C1}^{(t)}(r_0,r_k;J'M'JM) \tag{6.30}$$

with $J' = J\pm 1, J$

(5) $C2$**-term** The second potential coming from the C_t amplitude is written as

$$\mathcal{V}_k^{(t),C2}(\boldsymbol{r}_0-\boldsymbol{r}_k) = \bar{J}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}C_t(q)\boldsymbol{1}_0(\boldsymbol{\sigma}_k\cdot\hat{\boldsymbol{n}})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)}$$
$$= \sum_{J'M'}\sum_{JM} V_{C2}^{(t)}(r_0,r_k;J'M'JM)\left(\left[\mathrm{i}^{J'} Y_{J'}(\hat{\boldsymbol{r}}_0)\times\sigma_0^{(0)}\right]^{J'}_{M'}\right)\left(\left[\mathrm{i}^{J'} Y_{J'}(\hat{\boldsymbol{r}}_k)\times\sigma_k^{(1)}\right]^{J}_{M}\right)^{\dagger} \tag{6.31}$$

where

$$V_{C2}^{(t)}(r_0,r_k;J'M'JM) = \sum_{\mu}(J'M'\ 1\ \mu|JM)\hat{n}_{\mu}\bar{J}\frac{2}{\pi}\int_0^{\infty}C_t(q)j_{J'}(qr_0)j_{J'}(qr_k)q^2dq \tag{6.32}$$

Thus we get

$$V_{C2}^{(t)}(r_0,r;l's'J'M';lsJM) = \delta_{J'l'}\delta_{l'l}\delta_{s'0}\delta_{s1}\,V_{C2}^{(t)}(r_0,r_k;J'M'JM) \tag{6.33}$$

with $J = J'\pm 1, J'$.

6.3 Distorted Waves

6.3.1 Partial wave expansion of the distorted waves

(1) **Outgoing wave**

The distorted waves with the outgoing boundary condition $\chi^{(+)}_{\boldsymbol{k}m_s}(\boldsymbol{r})$ is expanded as [11]

$$
\begin{aligned}
\chi^{(+)}_{\boldsymbol{k}m_s}(\boldsymbol{r}) &= \sum_{m'_s} \chi^{(+)}_{m'_s m_s}(\boldsymbol{k}, \boldsymbol{r})\chi^s_{m'_s} \\
&= \sum_{m'_s} \frac{4\pi}{kr} \sum_{lmm'} \sum_{jm_j} (lmsm_s|jm_j) Y^*_{lm}(\hat{\boldsymbol{k}})(lm'sm'_s|jm_j) Y_{lm'}(\hat{\boldsymbol{r}}) \mathrm{i}^l e^{\mathrm{i}\sigma_l} u^{(+)}_{lj}(k,r)\chi^s_{m'_s} \\
&= \frac{4\pi}{kr} \sum_{lm} \sum_{jm_j} (lmsm_s|jm_j) Y^*_{lm}(\hat{\boldsymbol{k}}) e^{\mathrm{i}\sigma_l} u^{(+)}_{lj}(k,r) \left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}) \times \chi^s\right]^j_{m_j}
\end{aligned}
\tag{6.34}
$$

where $\chi^s_{m_s}$ ($s = \frac{1}{2}$) is the nucleon spin state vector with the z-projection m_s, and σ_l is the Coulomb phase shift. The radial part $u^{(+)}_{lj}(k,r)$ has the asymptotic behavior

$$
u^{(+)}_{lj}(k,r) \sim e^{i\delta_{lj}} \sin(kr - \eta_{\mathrm{C}} \ln(2kr) - \frac{l\pi}{2} + \sigma_l + \delta_{lj}), \tag{6.35}
$$

where η_{C} is the Sommerfeld parameter and δ_{lj} is the nuclear phase shift.

(2) **Incoming wave**

The distorted waves with the incoming boundary condition $\chi^{(-)}_{\boldsymbol{k}m_s}(\boldsymbol{r})$ is written as

$$
\chi^{(-)}_{\boldsymbol{k}m_s}(\boldsymbol{r}) = \sum_{m'_s} \chi^{(-)}_{m'_s m_s}(\boldsymbol{k}, \boldsymbol{r})\chi^s_{m'_s} \tag{6.36}
$$

The time reversal invariance leads to the relation

$$
\chi^{(-)}_{m'_s m_s}(\boldsymbol{k}, \boldsymbol{r}) = (-1)^{s+m_s} \chi^{(+)*}_{-m'_s,-m_s}(-\boldsymbol{k}, \boldsymbol{r})(-1)^{s+m'_s} \tag{6.37}
$$

which gives the formula

$$
\chi^{(-)}_{\boldsymbol{k}m_s}(\boldsymbol{r}) = \frac{4\pi}{kr} \sum_{lm} \sum_{jm_j} (lmsm_s|jm_j) Y^*_{lm}(\hat{\boldsymbol{k}}) \left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}) \times \chi^s\right]^j_{m_j} e^{-\mathrm{i}\sigma_l} u^{(+)*}_{lj}(k,r) \tag{6.38}
$$

6.3.2 Non-relativistic optical model

In the non-relativistic optical model, the radial part $u^{(+)}_{lj}(k,r) = u^{(+)}_{lj}(\rho)$ is calculated by the Schödinger equation

$$
\frac{d^2 u_{lj}(\rho)}{d\rho^2} + \left\{ \left(1 - \frac{U_{lj}(r)}{E}\right) - \frac{l(l+1)}{\rho^2} \right\} u_{lj}(\rho) = 0, \qquad \rho = kr \tag{6.39}
$$

with the optical potential

$$
U_{lj}(r) = U_c(r) + U_{ls}(r)\left(j(j+1) - l(l+1) - s(s+1)\right) \tag{6.40}
$$

6.3.3 Dirac phenomenology

In this model, we start with the Dirac equation to describe NA elastic scatterings

$$[-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta\left(m_{\mathrm{N}}+U_s(r)\right)+U_0(r)]\,\Psi(\boldsymbol{r})=E_k\Psi(\boldsymbol{r}) \tag{6.41}$$

with

$$E_k=\sqrt{m_{\mathrm{N}}^2+k^2} \tag{6.42}$$

where U_s is the scalar potential and U_0 is the vector potential which includes the Coulomb potential. They are phenomenologically determined to reproduce the NA elastic scattering. Writing the Dirac spinor as

$$\Psi(\boldsymbol{r})=\begin{pmatrix}\psi(\boldsymbol{r})\\ \psi'(\boldsymbol{r})\end{pmatrix} \tag{6.43}$$

we identify the distorted wave (6.34) to the upper component, namely,

$$\chi^{(+)}_{\boldsymbol{k}m_s}(\boldsymbol{r})=\psi(\boldsymbol{r}) \tag{6.44}$$

Defining

$$\phi(\boldsymbol{r})=\frac{1}{\sqrt{B(r)}}\psi(\boldsymbol{r}) \tag{6.45}$$

with

$$B(r)=\frac{E+m_{\mathrm{N}}+U_s(r)-U_0(r)}{E+m_{\mathrm{N}}} \tag{6.46}$$

we can derive the Schrödinger equivalent equation for $\phi(\boldsymbol{r})$

$$\left[\nabla^2+k^2-2E\left(U_{\mathrm{cen}}(r)+U_{\mathrm{so}}(r)\boldsymbol{\sigma}\cdot\boldsymbol{l}\right)\right]\phi(\boldsymbol{r})=0 \tag{6.47}$$

where

$$U_{\mathrm{cen}}(r) = \frac{1}{2E}\left(2EU_0(r)+2m_{\mathrm{N}}U_s(r)-U_0^2(r)+U_s^2(r)+U_D(r)\right) \tag{6.48}$$

$$U_D(r) = -\frac{1}{2r^2B(r)}\frac{d}{dr}\left(r^2\frac{dB(r)}{dr}\right)+\frac{3}{4B^2(r)}\left(\frac{dB(r)}{dr}\right)^2 \tag{6.49}$$

$$U_{\mathrm{so}}(r) = -\frac{1}{2E}\frac{1}{rB(r)}\frac{dB(r)}{dr} \tag{6.50}$$

Thus we obtain

$$\chi^{(+)}_{\boldsymbol{k}m_s}(\boldsymbol{r})=\sqrt{B(r)}\phi(\boldsymbol{r}) \tag{6.51}$$

6.4 Outer impulse functions

Explicitly writing the notations i and f as $i=(\boldsymbol{k}_i m_{s_i}\mathrm{N})$, $f=(\boldsymbol{k}_f m_{s_f}\mathrm{N}')$ and factoring out the isospin part, we write the outer impulse function (6.7) as

$$S^{fi}_{lsJM,t\nu}(r)=\langle\mathrm{N}'|\tau^{(t)}_\nu|\mathrm{N}\rangle S^{(t),\mathrm{N}'\mathrm{N}}_{m_{s_f}m_{s_i}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;lsJM) \tag{6.52}$$

$$= \langle\mathrm{N}'|\tau^{(t)}_\nu|\mathrm{N}\rangle\sum_{l's'J'M'}\int r_0^2dr_0V^{(t)}(r_0,r;l's'J'M';lsJM)f^{\mathrm{N}'\mathrm{N},m_{s_f},m_{s_i}}_{l's'J'M'}(\boldsymbol{k}_f,\boldsymbol{k}_i;r_0) \tag{6.53}$$

with

$$
\begin{aligned}
&f^{\mathrm{N'N},m_{s_f},m_{s_i}}_{l's'J'M'}(\boldsymbol{k}_f,\boldsymbol{k}_i;r_0) \\
&= \int d\Omega_0 \, \left(\chi^{\mathrm{N'}(-)}_{\boldsymbol{k}_f m_{s_f}}(\boldsymbol{r}_0)\right)^\dagger \left[\mathrm{i}^{l'} Y_{l'}(\theta_0,\phi_0)\times\sigma_0^{(s')}\right]^{J'}_{M'} \chi^{\mathrm{N}(+)}_{\boldsymbol{k}_i m_{s_i}}(\boldsymbol{r}_0)
\end{aligned}
\tag{6.54}
$$

Inserting eqs.(6.34) and (6.38), we obtain

$$
f^{m_{s_f} m_{s_i}}_{l's'J'M'}(\boldsymbol{k}_i,\boldsymbol{k}_f;r_0) \tag{6.55}
$$

$$
\begin{aligned}
= \; & \frac{\sqrt{4\pi}}{k_i k_f r_0^2}\sqrt{(2J'+1)(2l'+1)}\langle s_f||\sigma^{(s')}||s_i\rangle \sum_{l_i j_i}\sum_{l_f j_f} i^{l_i+l'-l_f} e^{i(\sigma_{l_f}+\sigma_{l_i})} \\
& \times u^{\mathrm{N'}(+)}_{l_f j_f}(k_f,r_0) u^{\mathrm{N}(+)}_{l_i j_i}(k_i,r_0)(2l_i+1)\sqrt{(2j_i+1)(2l_f+1)}(l_i 0 l' 0|l_f 0) \\
& \times \left\{ \begin{array}{ccc} l_f & s_f & j_f \\ l_i & s_i & j_i \\ l' & s' & J' \end{array} \right\} (l_i 0 s_i m_{s_i}|j_i m_{s_i})(l_f m_f s_f m_{s_f}|j_f \; m_{s_i}+M') \\
& \times (j_i m_{s_i} J'M'|j_f m_{s_i}+M')(-)^{\frac{m_f+|m_f|}{2}}\sqrt{\frac{(l_f-|m_f|)!}{(l_f+|m_f|)!}} P^{|m_f|}_{l_f}(\cos\theta_{\mathrm{cm}}).
\end{aligned}
\tag{6.56}
$$

where $s_i = s_f = \frac{1}{2}$ and

$$
\langle\frac{1}{2}||\sigma^{(0)}||\frac{1}{2}\rangle = 2\sqrt{2}, \quad \langle\frac{1}{2}||\sigma^{(1)}||\frac{1}{2}\rangle = \sqrt{6} \tag{6.57}
$$

Noting the relations

$$
\theta_{\boldsymbol{k}_i} = 0, \quad \theta_{\boldsymbol{k}_f} = \theta_{\mathrm{cm}}, \quad \phi_{\boldsymbol{k}_f} = 0 \tag{6.58}
$$

in the $[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ coordinate system, we used the formulas

$$
Y_{l_i m_i}(\hat{\boldsymbol{k}}_i) = \delta_{m_i 0}\sqrt{\frac{2l_i+1}{4\pi}} \tag{6.59}
$$

and

$$
Y_{l_f m_f}(\hat{\boldsymbol{k}}_f) = Y_{l_f m_f}(\theta_{\mathrm{cm}},0) = (-1)^{\frac{m_f+|m_f|}{2}} \left\{ \frac{2l_f+1}{4\pi}\frac{(l_f-|m_f|)!}{(l_f+|m_f|)!} \right\}^{1/2} P^{m_f}_{l_f}(\cos\theta_{\mathrm{cm}}) \tag{6.60}
$$

7 Cross sections expressed by response functions

7.1 Response functions for the spin-isospin density operators

We introduce the response functions for the spin-isospin density operators (6.8) as

$$
R^{t't}_{l's'ls,J\nu}(r',r;\omega) \equiv \sum_{n\neq 0}\langle\Psi^0_{\mathrm{A}}|\rho_{l's'JM,t'\nu}(r')|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\rho^\dagger_{lsJM,t\nu}(r)|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \tag{7.1}
$$

They are diagonal with respect to J and M because we assume that the spin of Ψ^0_{A} is 0. They are also diagonal with respect to ν because of the charge conservation.

7.2 Response functions for the outer impulse functions

From eqs. (6.6) and (6.52), the NA T-matrix is written as

$$\begin{aligned}&\left[T_{n0}^{\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i)\right]_{m_{s_f}m_{s_i}}\\&=\sum_t\langle \mathrm{N'}|\tau_\nu^{(t)}|\mathrm{N}\rangle\sum_{lsJM}\int r^2dr\,S_{m_{s_f}m_{s_i}}^{(t),\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;lsJM)\langle\Phi_n|\rho_{lsJM,t\nu}^{\dagger}(r)|\Phi_0\rangle\end{aligned}\tag{7.2}$$

Then we introduce the response functions for the outer impulse functions as

$$\begin{aligned}\left[R_S^{\mathrm{N'N}}\right]_{m_{s_f}m_{s_i};m'_{s_f}m'_{s_i}}&\equiv\sum_n\left[T_{n0}^{\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i)\right]_{m_{s_f}m_{s_i}}\left[T_{n0}^{\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i)\right]^{\dagger}_{m'_{s_f}m'_{s_i}}\delta(\omega-\omega_n)\\&=\sum_{tt'}\langle \mathrm{N'}|\tau_\nu^{(t')}|\mathrm{N}\rangle^{\dagger}\langle \mathrm{N'}|\tau_\nu^{(t)}|\mathrm{N}\rangle\sum_{JM}\sum_{l's'}\sum_{ls}\int r^2dr\int r'^2dr'\\&\times\left(S_{m_{s_f}m_{s_i}}^{(t'),\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;l's'JM)\right)^{\dagger}R_{l's'ls,J\nu}^{t't}(r',r;\omega)\,S_{m_{s_f}m_{s_i}}^{(t),\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;lsJM)\end{aligned}\tag{7.3}$$

where ν is fixed by the reaction type (N, N').

The present version of CRDW can treat only the cases of $t=t'=1$, thus

$$\begin{aligned}\left[R_S^{\mathrm{N'N}}\right]_{m_{s_f}m_{s_i};m'_{s_f}m'_{s_i}}&=\left|\langle \mathrm{N'}|\tau_\nu^{(1)}|\mathrm{N}\rangle\right|^2\sum_{JM}\sum_{l's'}\sum_{ls}\int r^2dr\int r'^2dr'\\&\times\left(S_{m_{s_f}m_{s_i}}^{(1),\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;l's'JM)\right)^{\dagger}R_{l's'ls,J\nu}^{11}(r',r;\omega)\,S_{m_{s_f}m_{s_i}}^{(1),\mathrm{N'N}}(\boldsymbol{k}_f,\boldsymbol{k}_i;r;lsJM)\end{aligned}\tag{7.4}$$

7.3 Observables in terms of the response functions

Now we can express the cross sections and the spin observables in terms of the response functions for the outer impulse functions.

7.3.1 Cross sections

The unpolarized cross sections are expressed from eq. (3.26) as

$$I^{\mathrm{N'N}}(\theta_{\mathrm{cm}},\omega_{\mathrm{cm}})=\frac{K}{2}\sum_{m_{s_f}m_{s_i}}\left[R_S^{\mathrm{N'N}}\right]_{m_{s_f}m_{s_i};m_{s_f}m_{s_i}}\tag{7.5}$$

7.3.2 Spin observables

The polarization P_y $(\times I^{\mathrm{N'N}})$ and the analyzing power $A_y(\times I^{\mathrm{N'N}})$ are expressed from eq. (3.32) as

$$I^{\mathrm{N'N}}P_y = K\mathrm{Im}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;-+}+\left[R_S^{\mathrm{N'N}}\right]_{+-;--}\right),\tag{7.6}$$

$$I^{\mathrm{N'N}}A_y = K\mathrm{Im}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;+-}+\left[R_S^{\mathrm{N'N}}\right]_{-+;--}\right)\tag{7.7}$$

The polarization transfer coefficients $D_{ij}(\times I^{\mathrm{N'N}})$ are expressed from eq. (3.38) as

$$I^{\mathrm{N'N}} D_{xx} = K\mathrm{Re}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;--} + \left[R_S^{\mathrm{N'N}}\right]_{+-;-+}\right) \tag{7.8}$$

$$I^{\mathrm{N'N}} D_{xz} = K\mathrm{Re}\left(\left[R_S^{\mathrm{N'N}}\right]_{+-;++} - \left[R_S^{\mathrm{N'N}}\right]_{--;-+}\right) \tag{7.9}$$

$$I^{\mathrm{N'N}} D_{yy} = K\mathrm{Re}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;--} - \left[R_S^{\mathrm{N'N}}\right]_{+-;-+}\right) \tag{7.10}$$

$$I^{\mathrm{N'N}} D_{zx} = K\mathrm{Re}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;-+} - \left[R_S^{\mathrm{N'N}}\right]_{--;+-}\right) \tag{7.11}$$

$$I^{\mathrm{N'N}} D_{zz} = \frac{K}{2}\left(\left[R_S^{\mathrm{N'N}}\right]_{++;++} - \left[R_S^{\mathrm{N'N}}\right]_{+-;+-} - \left[R_S^{\mathrm{N'N}}\right]_{-+;-+} + \left[R_S^{\mathrm{N'N}}\right]_{--;--}\right) \tag{7.12}$$

where the suffices $\pm$ denote $2m_s$.

8 Generalization to include Δ isobar

We make the following generalization to include the Δ isobar degrees of freedom.

8.1 Spin and isospin operators

The spin operators $\sigma^{(s)}$ are generalized as

$$\sigma_\mu^{(s)} \longrightarrow \sigma_\mu^{(s),ab} = \begin{cases} \sigma^{(0),ab} = \delta_{ab}\mathbf{1} \\ \sigma_\mu^{(1),\mathrm{NN}} = \sigma_\mu \\ \sigma_\mu^{(1),\Delta\mathrm{N}} = S_\mu \\ \sigma_\mu^{(1),\mathrm{N}\Delta} = (-1)^\mu (S_{-\mu})^\dagger \end{cases} \tag{8.1}$$

where $a, b = \mathrm{N}$ or Δ, and S_μ is the spin transition operators from N to Δ defined as

$$\langle m_\Delta | S_\mu | m_\mathrm{N} \rangle \equiv (\frac{1}{2} m_\mathrm{N} 1 \mu | \frac{3}{2} m_\Delta) \tag{8.2}$$

Similarly the isospin operators $\tau^{(t)}$ are generalized as

$$\tau_\mu^{(t)} \longrightarrow \tau_\mu^{(t),ab} \tag{8.3}$$

by replacing σ_μ and S_μ by τ_ν and T_ν, the isospin transition operators from N to Δ.

8.2 Density operators and response functions

The spin-isospin density operators (6.8) are generalized as

$$\rho_{lsJM,t\nu}^{ab}(r) \equiv \sum_k \tau_{\nu,k}^{(t),ab} \left[\mathrm{i}^l Y_l(\theta_k, \phi_k) \times \sigma_k^{(s),ab}\right]_M^J \frac{\delta(r - r_k)}{r r_k} \tag{8.4}$$

and their response functions (7.1) as

$$R_{l's'lsJ\nu}^{abcd,t't}(r', r; \omega) \equiv \sum_n \langle \Psi_\mathrm{A}^0 | \rho_{l's'JM,t'\nu}^{ab}(r') | \Psi_\mathrm{X}^n \rangle \langle \Psi_\mathrm{X}^n | \left(\rho_{lsJM,t\nu}^{cd}(r)\right)^\dagger | \Psi_\mathrm{A}^0 \rangle \delta(\omega - \omega_n) \tag{8.5}$$

8.3 NN t-matrix

We generalize the NN t-matrix to include NΔ transitions. Since information is very limited, we take a simple generalization of the isovector and spin-vector parts of the free NN t-matrix (4.10)

$$(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k) \Big[B_1 (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) + E_1 (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) + F_1 (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) \Big] \tag{8.6}$$

into the form of

$$\begin{aligned}
& (\boldsymbol{\tau}_0^{(1),\mathrm{NN}} \cdot \boldsymbol{\tau}_k^{(1),\mathrm{NN}}) \Big[B_1 (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) \\
& \qquad + E_1 (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) + F_1 (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) \Big] \\
+ \; & (\boldsymbol{\tau}_0^{(1),\mathrm{NN}} \cdot \boldsymbol{\tau}_k^{(1),\mathrm{N\Delta}}) \Big[B_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{N\Delta}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) \\
& \qquad + E_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{N\Delta}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) + F_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{N\Delta}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) \Big] \\
+ \; & (\boldsymbol{\tau}_0^{(1),\mathrm{NN}} \cdot \boldsymbol{\tau}_k^{(1),\mathrm{\Delta N}}) \Big[B_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{\Delta N}} \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) \\
& \qquad + E_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{\Delta N}} \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) + F_1' (\boldsymbol{\sigma}_0^{(1),\mathrm{NN}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k^{(1),\mathrm{N\Delta}} \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) \Big]
\end{aligned} \tag{8.7}$$

Considering the one-pion and one-rho meson exchange processes, we assume

$$E_1' = \frac{f_{\pi\mathrm{N\Delta}}}{f_{\pi\mathrm{NN}}} E_1, \qquad B_1' = \frac{f_{\rho\mathrm{N\Delta}}}{f_{\rho\mathrm{NN}}} B_1, \qquad F_1' = \frac{f_{\rho\mathrm{N\Delta}}}{f_{\rho\mathrm{NN}}} F_1 \tag{8.8}$$

where $f_{\pi\mathrm{NN}}$ and $f_{\pi\mathrm{N\Delta}}$ are πNN and πNΔ coupling constants, and $f_{\rho\mathrm{NN}}$ and $f_{\rho\mathrm{N\Delta}}$ are ρNN and ρNΔ coupling constants, respectively.

9 Polarization propagators

To calculate the response functions, we introduce the polarization propagators, for which RPA or TDA equations are developed.

9.1 Polarization propagators

The polarization propagators are defined by the sum of the forward and backward polarization propagators as

$$\Pi^{abcd,t't}_{l's'lsJ\nu}(r', r; \omega) \equiv \Pi^{\mathrm{FW},abcd,t't}_{l's'lsJ\nu}(r', r; \omega) + \Pi^{\mathrm{BK},abcd,t't}_{l's'lsJ\nu}(r', r; \omega) \tag{9.1}$$

with

$$\Pi^{\mathrm{FW},abcd,t't}_{l's'lsJ\nu}(r', r; \omega) \equiv \langle \Psi^0_{\mathrm{A}} | \rho^{ab}_{l's'JM,t'\nu}(r') \frac{1}{\omega - (H_A - \mathcal{E}^0_{\mathrm{A}}) + \mathrm{i}\eta} \left(\rho^{cd}_{lsJM,t\nu}(r) \right)^\dagger | \Psi^0_{\mathrm{A}} \rangle \tag{9.2}$$

$$\Pi^{\mathrm{BK},abcd,t't}_{l's'lsJ\nu}(r', r; \omega) \equiv \langle \Psi^0_{\mathrm{A}} | \left(\rho^{cd}_{lsJM,t\nu}(r) \right)^\dagger \frac{1}{-\omega - (H_A - \mathcal{E}^0_{\mathrm{A}}) + \mathrm{i}\eta} \rho^{ab}_{l's'JM,t'\nu}(r') | \Psi^0_{\mathrm{A}} \rangle \tag{9.3}$$

The response functions (8.5) are given by

$$R^{abcd,t't}_{l's'lsJ\nu}(r', r; \omega) = -\frac{1}{\pi} \mathrm{Im}\, \Pi^{abcd,t't}_{l's'lsJ\nu}(r', r; \omega) \tag{9.4}$$

9.2 Mean field approximation

We start with the mean field approximation.

9.2.1 Hamiltonian

In this approximation the intrinsic Hamiltonian $H_{\rm A}$ of eq. (3.21) is replaced by the mean field Hamiltonian H_0

$$H_A \longrightarrow H_0 = \sum_k^A \left(\hat{h}_k^n \oplus \hat{h}_k^p \oplus \hat{h}_k^{\Delta^-} \oplus \hat{h}_k^{\Delta^0} \oplus \hat{h}_k^{\Delta^+} \oplus \hat{h}_k^{\Delta^{++}}\right) - T_{\rm c.m.} \tag{9.5}$$

where $\hat{h}^\alpha$ is the single particle Hamiltonian of the particle $\alpha(= n, p, \Delta^-, \Delta^0, \Delta^+, \Delta^{++})$. We define the single particle states and their energies by

$$h^\alpha|h_\alpha\rangle = \epsilon_h^\alpha|h_\alpha\rangle, \quad \text{(for occupied states, } \alpha = n, p) \tag{9.6}$$

$$h^\alpha|p_\alpha\rangle = \epsilon_p^\alpha|p_\alpha\rangle, \quad \text{(for unoccupied states, } \alpha = n, p, \Delta^-, \Delta^0, \Delta^+, \Delta^{++}) \tag{9.7}$$

We write the ground state of the target A its energy of H_0 by $\Phi_{\rm A}^0$ and $\mathcal{E}_{\rm A}^{0(0)}$, thus

$$H_0\Phi_{\rm A}^0 = \mathcal{E}_{\rm A}^{0,(0)}\Phi_{\rm A}^0 \tag{9.8}$$

We assumed that the ground state does not include Δ and its spin $J_0 = 0$.

9.2.2 Free polarization propagator

(1) **Definitions**

We call the polarization propagator in this approximation is the *free (unperturbed) polarization propagator*. It is written as

$$\Pi_{l's'lsJ\nu}^{(0),abcd,t't}(r',r;\omega) \equiv \Pi_{l's'lsJ\nu}^{{\rm FW},(0),abcd,t't}(r',r;\omega) + \Pi_{l's'lsJ\nu}^{{\rm BK},(0),abcd,t't}(r',r;\omega) \tag{9.9}$$

with

$$\Pi_{l's'lsJ\nu}^{{\rm FW},(0),abcd,t't}(r',r;\omega) \equiv \langle\Phi_{\rm A}^0|\rho_{l's'JM,t'\nu}^{ab}(r')\frac{1}{\omega - (H_0 - \mathcal{E}_{\rm A}^{0,(0)}) + {\rm i}\eta}\left(\rho_{lsJM,t\nu}^{cd}(r)\right)^\dagger|\Phi_{\rm A}^0\rangle \tag{9.10}$$

$$\Pi_{l's'lsJ\nu}^{{\rm BK},(0),abcd,t't}(r',r;\omega) \equiv \langle\Phi_{\rm A}^0|\left(\rho_{lsJM,t\nu}^{cd}(r)\right)^\dagger\frac{1}{-\omega - (H_0 - \mathcal{E}_{\rm A}^{0,(0)}) + {\rm i}\eta}\rho_{l's'JM,t'\nu}^{ab}(r')|\Phi_{\rm A}^0\rangle \tag{9.11}$$

Since $\rho^{ab}|\Phi_{\rm A}^0\rangle$ and $\left(\rho^{ab}\right)^\dagger|\Phi_{\rm A}^0\rangle$ are the sum of 1-N(Δ)-particle-1-hole states, and the propagators $(\pm\omega - (H_0 - \mathcal{E}_{\rm A}^{0,(0)}) + {\rm i}\,\eta)^{-1}$ are diagonal with respect to these states, we can write

$$\Pi_{l's'lsJ\nu}^{(0),abcd,t't}(r',r;\omega) = \delta_{ac}\delta_{bd}\Pi_{l's'lsJ\nu}^{(0),ab,t't}(r',r;\omega) \tag{9.12}$$

$$\Pi_{l's'lsJ\nu}^{(0),{\rm FW},abcd,t't}(r',r;\omega) = \delta_{ac}\delta_{bd}\Pi_{l's'lsJ\nu}^{(0),{\rm FW}ab,t't}(r',r;\omega) \tag{9.13}$$

$$\Pi_{l's'lsJ\nu}^{(0),{\rm BK},abcd,t't}(r',r;\omega) = \delta_{ac}\delta_{bd}\Pi_{l's'lsJ\nu}^{(0),{\rm BK},ab,t't}(r',r;\omega) \tag{9.14}$$

thus

$$\Pi_{l's'lsJ\nu}^{(0),ab,t't}(r',r;\omega) = \Pi_{l's'lsJ\nu}^{{\rm FW},(0),ab,t't}(r',r;\omega) + \Pi_{l's'lsJ\nu}^{{\rm BK},(0),ab,t't}(r',r;\omega) \tag{9.15}$$

with

$$\Pi^{\mathrm{FW},(0),ab,t't}_{l's'lsJ\nu}(r',r;\omega) = \langle\Phi^0_{\mathrm{A}}|\rho^{ab}_{l's'JM,t'\nu}(r')\ G^{(0)}_{ph}(\omega)\left(\rho^{ab}_{lsJM,t\nu}(r)\right)^{\dagger}|\Phi^0_{\mathrm{A}}\rangle \tag{9.16}$$

$$\Pi^{\mathrm{BK},(0),ab,t't}_{l's'lsJ\nu}(r',r;\omega) = \langle\Phi^0_{\mathrm{A}}|\left(\rho^{ab}_{lsJM,t\nu}(r)\right)^{\dagger} G^{(0)}_{ph}(\omega)\ \rho^{ab}_{l's'JM,t'\nu}(r')\ |\Phi^0_{\mathrm{A}}\rangle \tag{9.17}$$

where $G^{(0)}_{ph}(\omega)$ is the *free ph Green's function* defined as

$$G^{(0)}_{ph}(\omega) \equiv \sum_{\alpha\alpha'}\sum_{h_{\alpha'}}\sum_{p_\alpha}|h^{-1}_{\alpha'}p_\alpha\rangle\frac{1}{\omega-(\epsilon^\alpha_p-\epsilon^{\alpha'}_h)+\mathrm{i}\,\eta}\langle h^{-1}_{\alpha'}p_\alpha| \tag{9.18}$$

(2) **Free ph Green's function**

Noting that $h_{\alpha'}$ runs over finite number of discrete occupied states, but p_α runs over infinite continuum states, we rewrite it as

$$G^{(0)}_{ph}(\omega) = \sum_{\alpha'\alpha} G^{(0),\alpha'\alpha}_{ph}(\omega) \tag{9.19}$$

$$G^{(0),\alpha'\alpha}_{ph}(\omega) = \sum_{h_{\alpha'}}|h^{-1}_{\alpha'}\rangle g^\alpha_p(\omega+\epsilon^{\alpha'}_h)\langle h^{-1}_{\alpha'}| \tag{9.20}$$

where

$$g^\alpha_p(\epsilon) = \sum_{p_\alpha}|p_\alpha\rangle\frac{1}{\epsilon-\epsilon^\alpha_p+\mathrm{i}\,\eta}\langle p_\alpha| = \sum_{p_\alpha}|p_\alpha\rangle\langle p_\alpha|\frac{1}{\epsilon-h^\alpha+\mathrm{i}\,\eta}|p_\alpha\rangle\langle p_\alpha| \tag{9.21}$$

We call $g^\alpha_p(\epsilon)$ the *particle Green's function.*

(3) **Grouping**

Since Φ^0_{A} does not contain Δ, we get the relations

$$\Pi^{(0),\Delta\Delta,t't} = 0, \qquad \Pi^{\mathrm{FW},(0),\Delta\mathrm{N},t't} = 0, \qquad \Pi^{\mathrm{BK},(0),\mathrm{N}\Delta,t't} = 0 \tag{9.22}$$

For later use, we introduce the grouped quantities

$$\Pi^{(0),[\mathrm{N}],t't} \equiv \Pi^{(0),\mathrm{NN},t't} \tag{9.23}$$

$$\Pi^{(0),[\Delta],t't} \equiv \Pi^{(0),\mathrm{N}\Delta,t't} + \Pi^{(0),\Delta\mathrm{N},t't} = \Pi^{\mathrm{FW},(0),\mathrm{N}\Delta,t't} + \Pi^{\mathrm{BK},(0),\Delta\mathrm{N},t't} \tag{9.24}$$

In eqs. (9.22) - (9.24), the suffices $(l's'lsJ\nu)$ and the arguments $(r',r;\omega)$ are suppressed.

9.3 Polarization propagator with nuclear correlations

We consider nuclear correlations only for the isovector ($t'=t=1$) and spin diagonal ($s'=s$) polarization propagators. So we simplify the notations as

$$\Pi^{abcd}_{l'lsJ\nu} \equiv \Pi^{abcd,t'=t=1}_{l's'=slsJ\nu}, \qquad \Pi^{(0),ab}_{l'lsJ\nu} \equiv \Pi^{(0),ab,t'=t=1}_{l's'=slsJ\nu}, \qquad \Pi^{(0),[a]}_{l'lsJ\nu} \equiv \Pi^{(0),[a],t'=t=1}_{l's'=slsJ\nu} \tag{9.25}$$

where the arguments $(r',r;\omega)$ are suppressed.

We take into account the nuclear correlations through RPA or TDA with the effective N(Δ) particle-hole interaction in the isovector channels.

$$V^{ph} = \sum_{k'<k} V^{ph}_{k'k}(\boldsymbol{r}_{k'}-\boldsymbol{r}_k;\omega) \tag{9.26}$$

which is expressed in the angular momentum representation as

$$\begin{aligned} V^{ph}_{k'k}(\boldsymbol{r}_{k'}-\boldsymbol{r}_k;\omega) &= \sum_{abcd}\sum_{l'ls}\sum_{JM\nu}\left(\tau^{(1),ab}_{\nu,k'}\left[\mathrm{i}^{l'}Y_{l'}(\theta_{k'},\phi_{k'})\times\sigma^{(s),ab}_{k'}\right]^J_M\right)^{\dagger} \\ &\times\ W^{abcd}_{l'lsJ}(r_{k'},r_k;\omega)\left(\tau^{(1),cd}_{\nu,k}\left[\mathrm{i}^{l}Y_{l}(\theta_{k},\phi_{k})\times\sigma^{(s),cd}_{k}\right]^J_M\right) \end{aligned} \tag{9.27}$$

9.3.1 RPA and TDA equations

In RPA with the ring approximation, the polarization propagators are given by the solution of the RPA equations

$$\Pi^{\mathrm{RPA},abcd}_{l'lsJ\nu}(r',r;\omega) = \delta_{ac}\delta_{bd}\Pi^{(0),ab}_{l'lsJ\nu}(r',r;\omega)$$
$$+\sum_{a'b'}\sum_{l_1l_2}\int_0^\infty r_1^2dr_1\int_0^\infty r_2^2dr_2\Pi^{(0),ab}_{l'l_1sJ\nu}(r',r_1;\omega)W^{aba'b'}_{l_1l_2sJ}(r_1,r_2;\omega)\Pi^{\mathrm{RPA},a'b'cd}_{l_2lsJ\nu}(r_2,r;\omega) \tag{9.28}$$

In TDA only the forward propagations are taken in to account, thus the TDA polarization propagators are given by the solution of the TDA equations

$$\Pi^{\mathrm{TDA},abcd}_{l'lsJ\nu}(r',r;\omega) = \delta_{ac}\delta_{bd}\Pi^{\mathrm{FW},(0),ab}_{l'lsJ\nu}(r',r;\omega)$$
$$+\sum_{a'b'}\sum_{l_1l_2}\int_0^\infty r_1^2dr_1\int_0^\infty r_2^2dr_2\Pi^{\mathrm{FW},(0),ab}_{l'l_1sJ\nu}(r',r_1;\omega)W^{aba'b'}_{l_1l_2sJ}(r_1,r_2;\omega)\Pi^{\mathrm{TDA},a'b'cd}_{l_2lsJ\nu}(r_2,r;\omega) \tag{9.29}$$

9.3.2 Simplification of the RPA and TDA equations

Noting the symmetries of W^{abcd} with respect to a,b,c,d, we introduce the notations

$$W^{[\mathrm{NN}]} \equiv W^{\mathrm{NNNN}}, \tag{9.30}$$
$$W^{[\mathrm{N\Delta}]} \equiv W^{\mathrm{NNN\Delta}} = W^{\mathrm{NN\Delta N}} = W^{\mathrm{\Delta NNN}} = W^{\mathrm{N\Delta NN}} \tag{9.31}$$
$$W^{[\Delta\Delta]} \equiv W^{\mathrm{N\Delta N\Delta}} = W^{\mathrm{N\Delta\Delta N}} = W^{\mathrm{\Delta N\Delta N}} = W^{\mathrm{\Delta NN\Delta}} \tag{9.32}$$

and the grouped polarization propagators as

$$\Pi^{[\mathrm{NN}]} \equiv \Pi^{\mathrm{NNNN}}, \tag{9.33}$$
$$\Pi^{[\mathrm{N\Delta}]} \equiv \Pi^{\mathrm{NNN\Delta}} + \Pi^{\mathrm{NN\Delta N}} \tag{9.34}$$
$$\Pi^{[\mathrm{\Delta N}]} \equiv \Pi^{\mathrm{\Delta NNN}} + \Pi^{\mathrm{N\Delta NN}} \tag{9.35}$$
$$\Pi^{[\Delta\Delta]} \equiv \Pi^{\mathrm{N\Delta N\Delta}} + \Pi^{\mathrm{N\Delta\Delta N}} + \Pi^{\mathrm{\Delta N\Delta N}} + \Pi^{\mathrm{\Delta NN\Delta}} \tag{9.36}$$

where we suppressed the suffices and the arguments.

Then the RPA and TDA equations are reduced to simpler forms

$$\Pi^{\mathrm{RPA}[ab]}_{l'lsJ\nu}(r',r;\omega) = \delta_{ab}\Pi^{(0)[a]}_{l'lsJ\nu}(r',r;\omega)$$
$$+\sum_{c}\sum_{l_1l_2}\int r_1^2r_2^2dr_1dr_2\ \Pi^{(0)[a]}_{l'l_1s,J\nu}(r',r_1;\omega)W^{[ac]}_{l_1l_2sJ}(r_1,r_2;\omega)\Pi^{\mathrm{RPA}[cb]}_{l_2ls,J\nu}(r_2,r;\omega) \tag{9.37}$$
$$\Pi^{\mathrm{TDA}[ab]}_{l'lsJ\nu}(r',r;\omega) = \delta_{ab}\Pi^{\mathrm{FW}(0)[a]}_{l'lsJ\nu}(r',r;\omega)$$
$$+\sum_{c}\sum_{l_1l_2}\int r_1^2r_2^2dr_1dr_2\ \Pi^{\mathrm{FW}(0)[a]}_{l'l_1sJ\nu}(r',r_1;\omega)W^{[ac]}_{l_1l_2sJ}(r_1,r_2;\omega)\Pi^{\mathrm{TDA}[cb]}_{l_2lsJ\nu}(r_2,r;\omega) \tag{9.38}$$

9.3.3 Response functions

Correspondingly, we use the simplified notations for the response functions

$$R^{abcd}_{l'lsJ\nu} \equiv R^{abcd,t'=t=1}_{l's'=slsJ\nu} = -\frac{1}{\pi}\mathrm{Im}\,\Pi^{abcd}_{l'lsJ\nu}(r',r;\omega) \tag{9.39}$$
$$R^{(0),[\mathrm{N}]}_{l'lsJ\nu} \equiv R^{(0),\mathrm{NN}}_{l'lsJ\nu} = -\frac{1}{\pi}\mathrm{Im}\,\Pi^{(0),[\mathrm{N}]}_{l'lsJ\nu}(r',r;\omega) \tag{9.40}$$
$$R^{(0),[\Delta]}_{l'lsJ\nu} \equiv R^{(0),\mathrm{N\Delta}}_{l'lsJ\nu} + R^{(0),\mathrm{\Delta N}}_{l'lsJ\nu} = -\frac{1}{\pi}\mathrm{Im}\,\Pi^{(0),[\Delta]}_{l'lsJ\nu}(r',r;\omega) \tag{9.41}$$

10 Calculation of the free polarization propagator

The key ingredients to calculate the polarization propagators are the free polarization propagators and the effective ph interactions. In this section, we explain details about calculation of the free polarization propagators presented in eqs. (9.9) - (9.11).

10.1 Single particle states

10.1.1 Nucleon single particle states

The nucleon single particle states obey the Schrödinger equation

$$h^\alpha \phi^\alpha(\boldsymbol{r};\epsilon) = \left[-\frac{1}{2\mu_{\rm N}}\boldsymbol{\nabla}^2 + \int d\boldsymbol{r}' U_\alpha(\boldsymbol{r},\boldsymbol{r}')\right]\phi^\alpha(\boldsymbol{r}';\epsilon) = \epsilon\phi^\alpha(\boldsymbol{r};\epsilon), \quad (\alpha = n,p) \tag{10.1}$$

with the reduced mass

$$\mu_{\rm N} = \frac{A-1}{A}m_{\rm N} \tag{10.2}$$

where we use the average nucleon mass

$$m_{\rm N} = \frac{m_n + m_p}{2} \tag{10.3}$$

and shift the origin of the energy by $m_{\rm N}$.

Dealing with the non-locality by the effective mass approximation[11], we get the Schrödinger equation with the local potential

$$\left[-\frac{1}{2\mu_{\rm N}}\boldsymbol{\nabla}^2 + U_\alpha(\boldsymbol{r})\right]\phi^{({\rm L}),\alpha}(\boldsymbol{r};\epsilon) = \epsilon\phi^{({\rm L}),\alpha}(\boldsymbol{r};\epsilon) \tag{10.4}$$

for the modified wave function $\phi^{({\rm L}),\alpha}(\boldsymbol{r})$, which is defined by

$$\phi^\alpha(\boldsymbol{r}) = \sqrt{P_\alpha(r)}\phi^{({\rm L}),\alpha}(\boldsymbol{r}), \qquad P_\alpha(r) = \frac{\mu^*_\alpha(r)}{\mu_{\rm N}} \tag{10.5}$$

where $P_\alpha(r)$ is the Perey factor and $\mu^*_\alpha(r)$ is the radial dependent effective mass of the nucleon $\alpha(=n,p)$.

Since we do not have information about the non-local potential $U_\alpha(\boldsymbol{r},\boldsymbol{r}')$, we treat the local potential $U_\alpha(\boldsymbol{r})$ and the Perey factor $P_\alpha(r)$ phenomenologically in the form of

$$U_\alpha(\boldsymbol{r}) = -(V_0^\alpha + {\rm i}W_0^\alpha)f_{\rm WS}^{c,\alpha}(r) + \frac{1}{m_\pi^2}\frac{V_{ls}^\alpha}{r}\frac{df_{\rm WS}^{ls,\alpha}(r)}{dr}\boldsymbol{l}\cdot\boldsymbol{\sigma} + V_{\rm C}^\alpha(r) \tag{10.6}$$

$$P_\alpha(r) = 1 - b_\alpha f_{\rm WS}^{\mu,\alpha}(r) \tag{10.7}$$

where

$$f_{\rm WS}^{x,\alpha} = \frac{1}{1+\exp\left(\frac{r - r_0^{x,\alpha}A^{1/3}}{a^{x,\alpha}}\right)}, \qquad (x = c, ls, \mu). \tag{10.8}$$

and $V_{\rm C}^\alpha(r)$ is the Coulomb potential. It is assumed to have the form

$$V_{\rm C}^\alpha(r) = \begin{cases} \frac{Z_\alpha Z_c e^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right), & (r < R_c = r_c^\alpha A_{\rm c}^{1/3}) \\ \frac{Z_\alpha Z_c e^2}{r}, & (r \geq R_c) \end{cases} \tag{10.9}$$

where Z_α is the charge of the nucleon α and Z_c is the charge of the core depending on α.

In the angular momentum representation, the wave functions are written as

$$\phi^{(L),\alpha}(\boldsymbol{r};\epsilon) = \frac{u^\alpha_{lj}(r;\epsilon)}{r}\mathcal{Y}_{lsjm}(\theta,\phi) \tag{10.10}$$

for the unbound states and

$$\phi^{(L),\alpha}(\boldsymbol{r};\epsilon^\alpha_{nlj}) = \frac{u^\alpha_{nlj}(r)}{r}\mathcal{Y}_{lsjm}(\theta,\phi) \tag{10.11}$$

for the bound states where

$$\mathcal{Y}_{lsjm}(\theta,\phi) = \sum_{m_l,m_s}(lm_lsm_s|jm)Y_{lm_l}(\theta,\phi)\chi^s_{m_s} \tag{10.12}$$

with $s=\frac{1}{2}$.

The radial wave functions obey the equation

$$\Bigg[-\frac{1}{2\mu_{\mathrm{N}}}\frac{d^2}{dr^2}+\frac{1}{2\mu_{\mathrm{N}}}\frac{l(l+1)}{r^2}-(V^\alpha_0+\mathrm{i}W^\alpha_0)f^{c,\alpha}_{\mathrm{WS}}(r)$$
$$+\frac{1}{m^2_\pi}\frac{V^\alpha_{ls}}{r}\frac{df^{ls,\alpha}_{\mathrm{WS}}(r)}{dr}\left(j(j+1)-l(l+1)-s(s+1)\right)+V^\alpha_{\mathrm{C}}(r)\Bigg]u^\alpha_{lj}(r)=\epsilon u^\alpha_{lj}(r) \tag{10.13}$$

where $u^\alpha_{lj}(r)=u^\alpha_{lj}(r;\epsilon)$ for the unbound states, and $u^\alpha_{lj}(r)=u^\alpha_{nlj}(r)$ and $\epsilon=\epsilon^\alpha_{nlj}$ for the bound states. The normalization of the bound state wave functions is given by

$$\int_0^\infty |u^\alpha_{nlj}(r)|^2P_\alpha(r)dr = 1 \tag{10.14}$$

In CRDW,

(1) The parameters $V^\alpha_{ls}, b_\alpha, r^{x,\alpha}_0, a^{x,\alpha},\ (x=c, ls, \mu)$ are input data.

(2) The real potential depth V^α_0 is determined to reproduce the binding energy of the highest occupied level of the nucleon α. Calculation is carried out with $W^\alpha_0 = 0$.

(3) The imaginary potential depth W^α_0 for the unoccupied states is given as an input value, or determined from the spreading width for the given energy ϵ by the phenomenological formula.

$$W^\alpha_0 = \frac{1}{2}\gamma^\alpha(\epsilon) = \alpha^\alpha_{\mathrm{spw}}\left[\frac{(\epsilon-\epsilon^\alpha_{\mathrm{F}})^2}{(\epsilon-\epsilon^\alpha_{\mathrm{F}})^2+(\varepsilon^\alpha_0)^2}\right]\left[\frac{(\varepsilon^{\mathrm{N}}_1)^2}{(\epsilon-\epsilon^{\mathrm{N}}_{\mathrm{F}})^2+(\varepsilon^{\mathrm{N}}_1)^2}\right] \tag{10.15}$$

where $\epsilon^\alpha_{\mathrm{F}}$ is the Fermi energy,

$$\epsilon^\alpha_{\mathrm{F}} = \frac{1}{2}\left(\epsilon^\alpha \text{ of the highest occupied level} + \epsilon^\alpha \text{ of the lowest unoccupied level}\right) \tag{10.16}$$

(4) The wave functions and energies ϵ^α_{nlj} of the occupied states are calculated with $W^\alpha_0 = 0$.

(5) The energies ϵ^α_{nlj} of the occupied states is artificially modified to the complex energy $\tilde{\epsilon}^\alpha_{nlj}$ as

$$\epsilon^\alpha_{nlj} \longrightarrow \tilde{\epsilon}^\alpha_{nlj} = \epsilon^\alpha_{nlj} - \mathrm{i}\frac{1}{2}\gamma^\alpha(\epsilon_{nlj}) \tag{10.17}$$

where the spreading width is given by input data or by the formula (10.15).

We note that this treatment of the imaginary potentials and the spreading widths violates the orthogonality between the occupied and the unoccupied single particle states.

10.1.2 Δ single particle states

For the mean field of Δ, we do not consider the non-locality, thus the Schödinger equation for the Δ single particle state is given by

$$h^\alpha \phi^\alpha(\boldsymbol{r};\epsilon) = \left[-\frac{1}{2\mu_\Delta}\nabla^2 + U_\alpha(\boldsymbol{r}) + \Delta m\right]\phi^\alpha(\boldsymbol{r};\epsilon) = \epsilon\phi^\alpha(\boldsymbol{r};\epsilon),$$

where $\alpha = \Delta^-, \Delta^0, \Delta^+, \Delta^{++}$,

$$\mu_\Delta = \frac{(A-1)m_{\mathrm{N}}m_\Delta}{(A-1)m_{\mathrm{N}} + m_\Delta} \tag{10.18}$$

and

$$\Delta m = m_\Delta - m_{\mathrm{N}} \tag{10.19}$$

The potential $U_\alpha(\boldsymbol{r})$ is assumed to have the form

$$U_\alpha(\boldsymbol{r}) = -\left(V_0^\alpha + \mathrm{i}W_0^\alpha\right) f_{\mathrm{WS}}^{c,\alpha}(r) + \frac{1}{m_\pi^2}\frac{V_{ls}^\alpha}{r}\frac{df_{\mathrm{WS}}^{ls,\alpha}(r)}{dr}\,\boldsymbol{l}\cdot\boldsymbol{\sigma}^{(1),\Delta\Delta} + V_{\mathrm{C}}^\alpha(r) \tag{10.20}$$

In CRDW,

$$V_0^\alpha,\ W_0^\alpha,\ V_{ls}^\alpha,\ r_0^{x,\alpha},\ a^{x,\alpha}, \qquad (x = c, ls,\ \alpha = \Delta^-, \Delta^0, \Delta^+, \Delta^{++}) \tag{10.21}$$

are the input data.

In the angular momentum representation we write the wave functions as

$$\phi^\alpha(\boldsymbol{r};\epsilon) = \frac{u_{lj}^\alpha(r;\epsilon')}{r}\mathcal{Y}_{lsjm}(\theta,\phi) \tag{10.22}$$

where $s = \frac{3}{2}$ and

$$\epsilon' = \epsilon - \Delta m \tag{10.23}$$

The radial wave function $u_{lj}^\alpha(r;\epsilon')$ satisfies the same equation of (10.13).

10.1.3 Single particle Green's functions

For later use we introduce the single particle Green's functions

$$g^\alpha(\epsilon) = \frac{1}{\epsilon - h^\alpha + \mathrm{i}\,\eta} \tag{10.24}$$

Its coordinate representation it can be written as

$$g^\alpha(\boldsymbol{r}',\boldsymbol{r},\epsilon) = \langle\boldsymbol{r}'|\frac{1}{\epsilon - h^\alpha + \mathrm{i}\eta}|\boldsymbol{r}\rangle = \sum_{ljm}\mathcal{Y}_{lsjm}(\theta',\phi')\frac{g_{lj}^\alpha(r',r;\epsilon)}{r'r}\left(\mathcal{Y}_{lsjm}(\theta,\phi)\right)^\dagger \tag{10.25}$$

where $s = \frac{1}{2}$ for $\alpha = n, p$, and $s = \frac{3}{2}$ for $\alpha = \Delta^-, \Delta^0, \Delta^+, \Delta^{++}$.

The radial part $g_{lj}^\alpha(r',r;\epsilon)$ is given by

$$g_{lj}^\alpha(r',r;\epsilon) = \begin{cases} \frac{2\mu_{\mathrm{N}}}{W\left(f_{lj}^\alpha(r;\epsilon),h_{lj}^{(+),\alpha}(r;\epsilon)\right)} f_{lj}^\alpha(r_<;\epsilon)h_{lj}^{(+),\alpha}(r_>;\epsilon), & (\alpha = n,p) \\ \frac{2\mu_\Delta}{W\left(f_{lj}^\alpha(r;\epsilon'),h_{lj}^{(+),\alpha}(r;\epsilon')\right)} f_{lj}^\alpha(r_<;\epsilon')h_{lj}^{(+),\alpha}(r_>;\epsilon'), & (\alpha = \Delta\,\mathrm{s}) \end{cases} \tag{10.26}$$

where $r_{<(>)}$ denotes smaller (larger) one of r and r'. The radial wave functions $f_{lj}^\alpha(r;\epsilon)$ and $h_{lj}^{(+),\alpha}(r;\epsilon)$ are the regular and the outgoing singular solutions of eq. (10.13), respectively, and $W(f,h)$ is the Wronskian of the functions f and h,

10.2 Particle Green's function

10.2.1 Nucleon particle Green's function

We must note that the particle Green's functions (9.21) of the nucleon ($\alpha = n, p$) are different from the single nucleon Green's function (10.24) because the summation over p_α is limmited over the unoccupied states.

Furthermore we may choose different potentials for the occupied and the unoccupied states, thus the orthogonality between the occupied and the unoccupied states.

To cope with these problems, Izumoto and Mori[10] derived the formula which calculates the particle Green's function g_p^α from the single particle Green's function g^α

$$g_p^\alpha(\epsilon) = g^\alpha(\epsilon) - g^\alpha(\epsilon)\Gamma^\alpha \left(\Gamma^\alpha g^\alpha(\epsilon)\Gamma^\alpha\right)^{-1} \Gamma^\alpha g^\alpha(\epsilon) \tag{10.27}$$

with the projection operator onto the occupied states

$$\Gamma^\alpha = \sum_{h_\alpha} |h_\alpha\rangle\langle h_\alpha| \tag{10.28}$$

We call this method a *orthogonality condition model.*

In the angular momentum representation, we can write

$$\langle r'l's'j'm'|g_p^\alpha(\epsilon)|rlsjm\rangle = \delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m'm}\frac{g_{p,lj}^\alpha(r',r;\epsilon)}{r'r} \tag{10.29}$$

$$\langle r'l's'j'm'|\Gamma|rlsjm\rangle = \delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m'm}\frac{\Gamma_{lj}^\alpha(r',r)}{r'r} \tag{10.30}$$

where $s = \frac{1}{2}$, and

$$\Gamma_{lj}^\alpha(r',r) = \sum_{n\in \mathrm{occ}} u_{nlj}^\alpha(r')u_{nlj}^{\alpha *}(r) \tag{10.31}$$

From eq.(10.27), we get

$$g_{p,lj}^\alpha(\epsilon) = g_{lj}^\alpha(\epsilon) - g_{lj}^\alpha(\epsilon)\Gamma_{lj}^\alpha \left(\Gamma_{lj}^\alpha g_{lj}^\alpha(\epsilon)\Gamma_{lj}^\alpha\right)^{-1} \Gamma_{lj}^\alpha g_{lj}^\alpha(\epsilon), \qquad (\alpha = n, p) \tag{10.32}$$

10.2.2 Particle Green's function of Δ

There are no occupied states of Δ, we can identify the particle Green's functions of Δ with the single Δ Green's functions, e. g.

$$g_p^\alpha(\epsilon) = g^\alpha(\epsilon), \qquad (\alpha = \Delta^-, \Delta^0, \Delta^+, \Delta^{++}) \tag{10.33}$$

10.3 Matrix elements of the free *ph* Green's functions

The matrix elements of the (α', α) component of the free *ph* Green's functions (9.20) is given by

$$\begin{aligned}
&\langle (n_h l_h j_h m_h \alpha')^{-1}(r' l_p j_p m_p \alpha)|G_{ph}^{\alpha'\alpha}(\omega)|(n_h l_h j_h m_h \alpha')^{-1}(r l_p j_p m_p \alpha)\rangle \\
&= \langle r' l_p j_p m_p \alpha|\frac{1}{\omega - (h^\alpha - \tilde{\epsilon}_{n_h l_h j_h}^{\alpha'}) + \mathrm{i}\eta}|r l_p j_p m_p \alpha\rangle \\
&= \frac{1}{rr'}\begin{cases} g_{p,l_p j_p}^\alpha\left(r', r;\ \omega + \tilde{\epsilon}_{n_h l_h j_h}^{\alpha'}\right) & (\alpha = n, p) \\ g_{l_p j_p}^\alpha\left(r', r;\ \omega + \tilde{\epsilon}_{n_h l_h j_h}^{\alpha'} - \Delta m\right) & (\alpha = \Delta^-, \Delta^0, \Delta^+, \Delta^{++}) \end{cases}
\end{aligned} \tag{10.34}$$

where $\alpha' = n, p$, and $\tilde{\epsilon}_{n_h l_h j_h}^{\alpha'}$ is given by eq. (10.17). The non-diagonal elements with respect to $((n_h, l_h, j_h, m_h\alpha')^{-1}, l_p, j_p, m_p\alpha)$ all vanish.

10.4 Matrix elements of the transition density operators

Using the formulas (eq. (3B-25) of ref. [13])

$$\begin{aligned}
\langle (j_h^{-1} j_p) J || O^{(J)} || 0 \rangle &= (-1)^{j_h+j_p-J} \langle j_p || O^{(J)} || j_h \rangle && (10.35)\\
\langle 0 || O^{(J)} || (j_h^{-1} j_p) J \rangle &= -\langle j_h || O^{(J)} || j_p \rangle && (10.36)
\end{aligned}$$

we obtain the matrix elements of the transition density operators

$$\begin{aligned}
&\langle (n_h l_h j_h \alpha')^{-1} (r l_p j_p \alpha); JM | \rho^{ab}_{lsJM,t\nu}(r) | \Phi^0_{\rm A} \rangle \\
&\qquad = \delta_{b\rm N} \langle \alpha | \tau^{(t),a\rm N}_{\nu} | \alpha' \rangle (-1)^{j_p+j_h-J} B^{ab}_{lsJ}(p,h) \frac{u^{\alpha'}_{n_h l_h j_h}(r)}{r}, && (10.37)\\
&\langle \Phi^0_{\rm A} | \rho^{ab}_{lsJM,t\nu}(r) | (n_h l_h j_h \alpha')^{-1} (r l_p j_p \alpha); J - M \rangle \\
&\qquad = -\delta_{a\rm N} \langle \alpha' | \tau^{(t),{\rm N}b}_{\nu} | \alpha \rangle (-1)^{J-M} \mathcal{B}^{ab}_{lsJ}(h,p) \frac{u^{\alpha'}_{n_h l_h j_h}(r)}{r} && (10.38)
\end{aligned}$$

where $\alpha' = n, p,\ \alpha = n, p, \Delta^-, \Delta^0, \Delta^+, \Delta^{++}$, and

$$\mathcal{B}^{ab}_{lsJ}(x,y) \equiv \sqrt{(2j_x+1)(2j_y+1)} \left\{ \begin{array}{ccc} l_x & s_x & j_x \\ l_y & s_y & j_y \\ l & s & J \end{array} \right\} \langle l_x || {\rm i}^l Y_l(\hat{\boldsymbol{r}}) || l_y \rangle \langle s_x || \sigma^{(s),ab} || s_y \rangle \quad (10.39)$$

Here we used the formulas

$$\begin{aligned}
&\langle l_x s_x j_x || \left[{\rm i}^l Y_l(\hat{\boldsymbol{r}}) \times \sigma^{(s),ab} \right]^{(J)} || l_y s_y j_y \rangle \\
= &\sqrt{(2J+1)(2j_x+1)(2j_y+1)} \left\{ \begin{array}{ccc} l_x & s_x & j_x \\ l_y & s_y & j_y \\ l & s & J \end{array} \right\} \langle l_x || {\rm i}^l Y_l(\hat{\boldsymbol{r}}) || l_y \rangle \langle s_x || \sigma^{(s),ab} || s_y \rangle \quad (10.40)
\end{aligned}$$

The double bar matrix elements of the spin operators are given by

$$\begin{aligned}
&\langle \tfrac{1}{2} || \sigma^{(0),|\rm NN} || \tfrac{1}{2} \rangle = \sqrt{2}, \quad \langle \tfrac{1}{2} || \sigma^{(1),\rm NN} || \tfrac{1}{2} \rangle = \sqrt{6}, && (10.41)\\
&\langle \tfrac{3}{2} || \sigma^{(1),\Delta\rm N} || \tfrac{1}{2} \rangle = 2, \qquad \langle \tfrac{1}{2} || \sigma^{(1),\rm N\Delta} || \tfrac{3}{2} \rangle = -2, && (10.42)
\end{aligned}$$

and that of the orbital part is given by

$$\langle l_x || {\rm i}^l Y_l(\hat{\boldsymbol{r}}) || l_y \rangle = {\rm i}^{l_y - l_x + l} \sqrt{\frac{(2l+1)(2l_y+1)}{4\pi}} (l_y 0 l 0 | l_x 0). \quad (10.43)$$

which is real because $l_y - l_x + l$ =even.

The matrix elements of the isospin operators are given by

$$\begin{aligned}
&\langle n | \tau^{(0),\rm NN}_0 | n \rangle = 1, \qquad \langle p | \tau^{(0),\rm NN}_0 | p \rangle = 1, && (10.44)\\
&\langle n | \tau^{(1),\rm NN}_0 | n \rangle = 1, \qquad \langle p | \tau^{(1),\rm NN}_0 | p \rangle = -1 && (10.45)\\
&\langle n | \tau^{(1),\rm NN}_{+1} | p \rangle = -\sqrt{2}, \quad \langle p | \tau^{(1),\rm NN}_{-1} | n \rangle = \sqrt{2} && (10.46)\\
&\langle \Delta^0 | \tau^{(1),\Delta\rm N}_{+1} | p \rangle = \sqrt{\frac{1}{3}}, \quad \langle \Delta^+ | \tau^{(1),\Delta\rm N}_{0} | p \rangle = \sqrt{\frac{2}{3}}, \quad \langle \Delta^{++} | \tau^{(1),\Delta\rm N}_{-1} | p \rangle = 1 && (10.47)\\
&\langle \Delta^- | \tau^{(1),\Delta\rm N}_{+1} | n \rangle = 1, \quad \langle \Delta^0 | \tau^{(1),\Delta\rm N}_{0} | n \rangle = \sqrt{\frac{2}{3}}, \quad \langle \Delta^+ | \tau^{(1),\Delta\rm N}_{-1} | n \rangle = \sqrt{\frac{1}{3}} && (10.48)
\end{aligned}$$

10.5 Matrix elements of the free polarization propagators

From eqs. (9.16), (9.17), (9.23) and (9.24), the forward and backward free polarization propagators are written as

$$\Pi^{\mathrm{FW},(0),[\mathrm{N}],t't}_{l's'lsJ\nu}(r',r;\omega) = \langle\Phi^0_{\mathrm{A}}|\rho^{\mathrm{NN}}_{l's'JM,t'\nu}(r')G^{(0)}_{ph}(\omega)\left(\rho^{\mathrm{NN}}_{lsJM,t\nu}(r)\right)^\dagger|\Phi^0_{\mathrm{A}}\rangle \tag{10.49}$$

$$\Pi^{\mathrm{BK},(0),[\mathrm{N}],t't}_{l's'lsJ\nu}(r',r;\omega) = \langle\Phi^0_{\mathrm{A}}|\left(\rho^{\mathrm{NN}}_{lsJM,t\nu}(r)\right)^\dagger G^{(0)}_{ph}(-\omega)\rho^{\mathrm{NN}}_{l's'JM,t'\nu}(r')|\Phi^0_{\mathrm{A}}\rangle \tag{10.50}$$

$$\Pi^{\mathrm{FW},(0),[\Delta],t't}_{l's'lsJ\nu}(r',r;\omega) = \langle\Phi^0_{\mathrm{A}}|\rho^{\mathrm{N}\Delta}_{l's'JM,t'\nu}(r')G^{(0)}_{ph}(\omega)\left(\rho^{\mathrm{N}\Delta}_{lsJM,t\nu}(r)\right)^\dagger|\Phi^0_{\mathrm{A}}\rangle \tag{10.51}$$

$$\Pi^{\mathrm{BK},(0),[\Delta],t't}_{l's'lsJ\nu}(r',r;\omega) = \langle 0|\left(\rho^{\Delta\mathrm{N}}_{lsJM,t\nu}(r)\right)^\dagger G^{(0)}_{ph}(-\omega)\rho^{\Delta\mathrm{N}}_{l's'JM,t'\nu}(r')|\Phi^0_{\mathrm{A}}\rangle \tag{10.52}$$

By use of eqs. (9.19), (10.34), (10.37) and (10.38), the free forward polarization propagator (10.49) is given by

$$\begin{aligned}
&\Pi^{\mathrm{FW},(0),[\mathrm{N}],t't}_{l's'lsJ\nu}(r',r;\omega)\\
&= \sum_{\alpha=n,p}\sum_{\alpha'=n,p}\sum_{n_hl_hj_h}\sum_{l_pj_p}\langle\Phi^0_{\mathrm{A}}|\rho^{\mathrm{NN}}_{l's'JM,t'\nu}(r')|(n_hl_hj_h)^{-1}(r_pl_pj_p);J-M\rangle\\
&\times\langle(n_hl_hj_h)^{-1}(r'_pl_pj_p);J-M|G^{(0),\alpha'\alpha}_{ph}(\omega)|(n_hl_hj_h)^{-1}(r_pl_pj_p);J-M\rangle\\
&\times\langle(n_hl_hj_h)^{-1}(r_pl_pj_p);J-M|\left(\rho^{\mathrm{NN}}_{lsJM,t\nu}(r)\right)^\dagger|\Phi^0_{\mathrm{A}}\rangle\\
&= \sum_{\alpha=n,p}\sum_{\alpha'=n,p}\sum_{n_hl_hj_h}\sum_{l_pj_p}\langle\alpha'|\tau^{(t'),\mathrm{NN}}_\nu|\alpha\rangle\langle\alpha|(\tau^{(t),\mathrm{NN}}_\nu)^\dagger|\alpha'\rangle\mathcal{B}^{\mathrm{NN}}_{l's'J}(h,p)\mathcal{B}^{\mathrm{NN}}_{lsJ}(h,p)\\
&\times\frac{u^{\alpha'}_{n_hl_hj_h}(r')}{r'}\frac{g^\alpha_{p,l_pj_p}(r',r;\omega+\tilde\epsilon^{\alpha'}_{n_hl_hj_h})}{r'r}\frac{u^{\alpha'}_{n_hl_hj_h}(r)}{r}
\end{aligned}\tag{10.53}$$

Similarly we get

$$\begin{aligned}
&\Pi^{\mathrm{FW},(0),[\Delta],t't}_{l's'lsJ\nu}(r',r;\omega)\\
&= \delta_{t'1}\delta_{t1}\delta_{s'1}\delta_{s1}\sum_{\alpha'=n,p}\sum_{\alpha=\Delta'\mathrm{s}}\sum_{n_hl_hj_h}\sum_{l_pj_p}|\langle\alpha'|\tau^{(1),\mathrm{N}\Delta}_\nu|\alpha\rangle|^2\mathcal{B}^{\mathrm{N}\Delta}_{l'1J}(h,p)\mathcal{B}^{\mathrm{N}\Delta}_{l1J}(h,p)\\
&\times\frac{u^{\alpha'}_{n_hl_hj_h}(r')}{r'}\frac{g^\alpha_{l_pj_p}(r',r;\ \omega+\tilde\epsilon^{\alpha'}_{n_hl_hj_h}-\Delta m)}{r'r}\frac{u^{\alpha'}_{n_hl_hj_h}(r)}{r}
\end{aligned}\tag{10.54}$$

$$\begin{aligned}
&\Pi^{\mathrm{BK},(0),[\mathrm{N}],t't}_{l's'lsJ\nu}(r',r;\omega)\\
&= \sum_{\alpha=n,p}\sum_{\alpha'=n,p}\sum_{n_hl_hj_h}\sum_{l_pj_p}\langle\alpha'|(\tau^{(t),\mathrm{NN}}_\nu)^\dagger|\alpha\rangle\langle\alpha|\tau^{(t'),\mathrm{NN}}_\nu|\alpha'\rangle\mathcal{B}^{\mathrm{NN}}_{lsJ}(p,h)\mathcal{B}^{\mathrm{NN}}_{l's'J}(p,h)\\
&\times\frac{u^{\alpha'}_{n_hl_hj_h}(r)}{r}\frac{g^\alpha_{p,l_pj_p}(r,r';-\omega+\tilde\epsilon^{\alpha'}_{n_hl_hj_h})}{rr'}\frac{u^{\alpha'}_{n_hl_hj_h}(r')}{r'}
\end{aligned}\tag{10.55}$$

$$\begin{aligned}
&\Pi^{\mathrm{BK},(0),[\Delta],t't}_{l's'lsJ\nu}(r',r;\omega)\\
&= \delta_{t'1}\delta_{t1}\delta_{s'1}\delta_{s1}\sum_{\alpha'=n,p}\sum_{\alpha=\Delta'\mathrm{s}}\sum_{n_hl_hj_h}\sum_{l_pj_p}|\langle\alpha|\tau^{(1),\Delta\mathrm{N}}_\nu|\alpha'\rangle|^2\mathcal{B}^{\Delta\mathrm{N}}_{l1J}(p,h)\mathcal{B}^{\Delta\mathrm{N}}_{l'1J}(p,h)\\
&\times\frac{u^{\alpha'}_{n_hl_hj_h}(r)}{r}\frac{g^\alpha_{l_pj_p}(r,r';-\omega+\tilde\epsilon^{\alpha'}_{n_hl_hj_h}-\Delta m)}{r'r}\frac{u^{\alpha'}_{n_hl_hj_h}(r')}{r'}
\end{aligned}\tag{10.56}$$

11 Effective ph Interactions

The other key ingredient to calculate the polarization propagators is the effective ph interactions. In this section, we present their explicit expressions.

In CRDW, we only consider nuclear correlations for the isovector spin-scalar and isovector spin-vector modes. So we decompose the effective ph interaction into the spin-scalar and spin-vector parts as

$$V_{k'k}^{ph}(\boldsymbol{r}_{k'} - \boldsymbol{r}_k; \omega) = V_{k'k}^{ss}(\boldsymbol{r}_{k'} - \boldsymbol{r}_k; \omega) + V_{k'k}^{sv}(\boldsymbol{r}_{k'} - \boldsymbol{r}_k; \omega) \tag{11.1}$$

11.1 Spin-scalar part

Referring to the general form (9.27), $s = 0$, $l' = l = J$ and $a = b = c = d =$N for the spin-scalar part because the Δ does not involved. Noting the abbreviated notation (9.30), this part is expressed as

$$V_{12}^{ss}(\boldsymbol{r}_1 - \boldsymbol{r}_2; \omega) = \sum_{\nu} \sum_{lm} \left(\tau_{\nu,1}^{(1),\mathrm{NN}} \mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}}_1) \right)^\dagger W_{ll0l}^{[\mathrm{NN}]}(r_1, r_2; \omega) \left(\tau_{\nu,2}^{(1),\mathrm{NN}} \mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}}_2) \right) \tag{11.2}$$

In CRDW, we take the contact form

$$V_{12}^{ss}(\boldsymbol{r}_1 - \boldsymbol{r}_2; \omega) = (\boldsymbol{\tau_1} \cdot \boldsymbol{\tau_2}) V_\tau \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{11.3}$$

thus

$$W_{ll0l}^{[\mathrm{NN}]}(r_1, r_2; \omega) = V_\tau \frac{\delta(r_1 - r_2)}{r_1 r_2} \tag{11.4}$$

The strength V_τ is the input parameter.

11.2 Spin-vector part

Referring to eq. (9.27), the spin-vector part is expressed as

$$\begin{aligned} V_{12}^{sv}(\boldsymbol{r}_1 - \boldsymbol{r}_2; \omega) &= \sum_{abcd} \sum_{\nu} \sum_{l'l} \sum_{JM} \left(\tau_{\nu,1}^{(1),ab} \left[\mathrm{i}^{l'} Y_{l'}(\hat{\boldsymbol{r}}_1) \times \sigma_1^{(1),ab} \right]_M^J \right)^\dagger \\ &\times \; W_{l'l1J}^{abcd}(r_1, r_2; \omega) \left(\tau_{\nu,2}^{(1),cd} \left[\mathrm{i}^l Y_l(\hat{\boldsymbol{r}}_2) \times \sigma_2^{(1),cd} \right]_M^J \right) \end{aligned} \tag{11.5}$$

with $(-1)^l = (-1)^{l'}$.

In CRDW, this part is presented in the form of

$$V_{12}^{sv}(\boldsymbol{r}_1 - \boldsymbol{r}_2; \omega) = \int \frac{d\boldsymbol{q}^3}{(2\pi)^3} V_{12}^{sv}(\boldsymbol{q}, \omega) \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \tag{11.6}$$

and its momentum representation $V_{12}^{sv}(\boldsymbol{q}, \omega)$ is given by the input data.

It is decomposed into the spin-longitudinal and spin-transverse parts as

$$V_{12}^{sv}(\boldsymbol{q}, \omega) = V_{12}^{\mathrm{L}}(\boldsymbol{q}, \omega) + V_{12}^{\mathrm{T}}(\boldsymbol{q}, \omega) \tag{11.7}$$

which have the forms

$$\begin{aligned}
V_{12}^{\mathrm{L}}(\boldsymbol{q},\omega) &= W_{\mathrm{L}}^{\mathrm{NN}}(q,\omega)\,(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{q}}) \\
&+ W_{\mathrm{L}}^{\mathrm{N}\Delta}(q,\omega)\,[\{(\boldsymbol{\tau}_1\cdot\boldsymbol{T}_2)(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{q}})(\boldsymbol{S}_2\cdot\hat{\boldsymbol{q}})+(1\leftrightarrow 2)\}+\mathrm{h.c.}] \\
&+ W_{\mathrm{L}}^{\Delta\Delta}(q,\omega)\left[\left\{(\boldsymbol{T}_1\cdot\boldsymbol{T}_2)(\boldsymbol{S}_1\cdot\hat{\boldsymbol{q}})(\boldsymbol{S}_2\cdot\hat{\boldsymbol{q}})+(\boldsymbol{T}_1\cdot\boldsymbol{T}_2^{\dagger})(\boldsymbol{S}_1\cdot\hat{\boldsymbol{q}})(\boldsymbol{S}_2^{\dagger}\cdot\hat{\boldsymbol{q}})\right\}+\ \mathrm{h.c.}\right] \\
V_{12}^{\mathrm{T}}(\boldsymbol{q},\omega) &= W_{\mathrm{T}}^{\mathrm{NN}}(q,\omega)\,(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\times\hat{\boldsymbol{q}})(\boldsymbol{\sigma}_2\times\hat{\boldsymbol{q}}) \\
&+ W_{\mathrm{T}}^{\mathrm{N}\Delta}(q,\omega)\,[\{(\boldsymbol{\tau}_1\cdot\boldsymbol{T}_2)(\boldsymbol{\sigma}_1\times\hat{\boldsymbol{q}})(\boldsymbol{S}_2\times\hat{\boldsymbol{q}})+(1\leftrightarrow 2)\}+\mathrm{h.c.}] \\
&+ W_{\mathrm{T}}^{\Delta\Delta}(q,\omega)\left[\left\{(\boldsymbol{T}_1\cdot\boldsymbol{T}_2)(\boldsymbol{S}_1\times\hat{\boldsymbol{q}})(\boldsymbol{S}_2\times\hat{\boldsymbol{q}})+(\boldsymbol{T}_1\cdot\boldsymbol{T}_2^{\dagger})(\boldsymbol{S}_1\times\hat{\boldsymbol{q}})(\boldsymbol{S}_2^{\dagger}\times\hat{\boldsymbol{q}})\right\}\right. \\
&\qquad\qquad \left. +\ \mathrm{h.c.}\right] \qquad (11.8)
\end{aligned}$$

Defining

$$W_{l'lJ}^{\mathrm{L},ab}(r',r;\omega) \equiv \frac{2}{\pi}\int q^2 dq j_{l'}(qr')W_{\mathrm{L}}^{ab}(q;\omega)j_l(qr) \qquad (11.9)$$

$$W_{l'lJ}^{\mathrm{T},ab}(r',r;\omega) \equiv \frac{2}{\pi}\int q^2 dq j_{l'}(qr')W_{\mathrm{T}}^{ab}(q;\omega)j_l(qr) \qquad (11.10)$$

and noting the abbreviated form (9.30)-(9.32), we obtain

$$W_{l'ls=1J}^{[ab]}(r',r;\omega) = \delta_{l'l}\delta_{lJ}W_{llJ}^{\mathrm{T},ab}(r',r;\omega) + a_{Jl'}a_{Jl}\left(W_{l'lJ}^{\mathrm{L},ab}(r',r;\omega) - W_{l'lJ}^{\mathrm{T},ab}(r',r;\omega)\right) \qquad (11.11)$$

where we used the generalized form of the formula (6.20).

In CRDW the $\pi+\rho+g'+h'$ model is used for the isovector spin-vector ph interaction. It consists of the one-pion exchange term, the one-rho-meson exchange term, the contact spin-spin term (g' term), and the contact tensor term (h' term). The expression of $W_{\mathrm{L}}^{ab}(q,\omega)$ and $W_{\mathrm{T}}^{ab}(q,\omega)$ is given in CRDW Manual sect. 5.4.3.

12 Specific response functions

CRDW optionally presents the isovector spin-scalar response function $R_{\mathrm{S}}(q,\omega)$, the isovector spin-vector one $R_{\mathrm{V}}(q,\omega)$, the isovector spin-longitudinal one $R_{\mathrm{L}}(q,\omega)$, and the isovector spin-transverse one $R_{\mathrm{T}}(q,\omega)$.

12.1 Momentum representation

First we prepare the momentum representation of the general isovector spin-diagonal response functions.

12.1.1 Spin-isospin density operators

Generalizing eq.(6.9), we introduce the spin-isospin density operators in the N+Δ space

$$\rho_{\nu,\mu}^{ts,ab}(\boldsymbol{r}) = \sum_k \tau_{\nu,k}^{(t),ab}\sigma_{\mu,k}^{(s),ab}\delta(\boldsymbol{r}-\boldsymbol{r}_k) = \sum_{JM}\sum_{lm}(lms\mu|JM)\left(\mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}})\right)^*\rho_{lsJM,t\nu}^{ab}(r) \qquad (12.1)$$

where $a,b=\mathrm{N}$ or Δ and $\rho_{lsJM,t\nu}^{ab}(r)$ is defined in (8.4). Their momentum representation is given by

$$\rho_{\nu,\mu}^{ts,ab}(\boldsymbol{q}) = \int d\boldsymbol{r}^3\rho_{\nu,\mu}^{ts,ab}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} = \sum_{JM}\sum_{lm}(lms\mu|JM)(-1)^l Y_{lm}^*(\hat{\boldsymbol{q}})\rho_{lsJM,t\nu}^{ab}(q) \qquad (12.2)$$

with

$$\rho^{ab}_{lsJM,t\nu}(q) = 4\pi \int r^2 dr \rho^{ab}_{lsJM,t\nu}(r) j_l(qr) \tag{12.3}$$

12.1.2 Response functions

We define the isovector spin-diagonal response functions in the coordinate space as

$$R^{abcd}_{s\mu'\mu,\nu}(\boldsymbol{r}',\boldsymbol{r};\omega) \equiv \sum_n \langle\Psi^0_{\mathrm{A}}|\rho^{ab}_{s\mu',t=1\nu}(\boldsymbol{r}')|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left(\rho^{cd}_{s\mu,t=1\nu}(\boldsymbol{r})\right)^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \tag{12.4}$$

By use of $R^{abcd}_{l'lsJ\nu}(r',r;\omega)$ of eq. (8.5), it is expanded as

$$\begin{aligned} & R^{abcd}_{s\mu'\mu,\nu}(\boldsymbol{r}',\boldsymbol{r};\omega) \\ = & \sum_{JM}\sum_{l'm'}\sum_{lm}(l'm's\mu'|JM)(lms\mu|JM)\left(\mathrm{i}^{l'}Y_{l'm'}(\hat{\boldsymbol{r}}')\right)^* R^{abcd}_{l'lsJ\nu}(r',r;\omega)\left(\mathrm{i}^l Y_{lm}(\hat{\boldsymbol{r}})\right) \end{aligned} \tag{12.5}$$

and their momentum representations are given by

$$\begin{aligned} R^{abcd}_{s\mu'\mu,\nu}(\boldsymbol{q}',\boldsymbol{q};\omega) &= \int d\boldsymbol{r}'^3 \int d\boldsymbol{r}^3 \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}'} R^{abcd}_{s\mu'\mu,\nu}(\boldsymbol{r}',\boldsymbol{r};\omega)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \\ &= \sum_n \langle\Psi^0_{\mathrm{A}}|\rho^{ab}_{s\mu',1\nu}(\boldsymbol{q}')|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left(\rho^{cd}_{s\mu,1\nu}(\boldsymbol{q})\right)^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \quad (12.6) \\ &= \sum_{JM}\sum_{l'm'}\sum_{lm}(l'm's\mu'|JM)(lms\mu|JM)Y^*_{l'm'}(\hat{\boldsymbol{q}}')R^{abcd}_{l'lsJ\nu}(q',q;\omega)Y_{lm}(\hat{\boldsymbol{q}}) \quad (12.7) \end{aligned}$$

with

$$\begin{aligned} R^{abcd}_{l'lsJ\nu}(q',q;\omega) &= \sum_{n'}\langle\Psi^0_{\mathrm{A}}|\rho^{ab}_{l'sJM,1\nu}(q')|\Psi^{n'JM}_{\mathrm{X}}\rangle\langle\Psi^{n'JM}_{\mathrm{X}}|\left(\rho^{cd}_{lsJM,1\nu}(q)\right)^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \\ &= (4\pi)^2\int r'^2dr'\int r^2 dr j_{l'}(q'r')R^{abcd}_{l'lsJ\nu}(r',r;\omega)j_l(qr) \end{aligned} \tag{12.8}$$

where $(-1)^{l'+l}=1$ is used,

12.2 Specific isovector response functions

We consider the following specific isovector response functions.

12.2.1 Isovector spin-scalar mode

We consider the isovector spin-scalar transition operators

$$O_{\nu,\mathrm{S}}(\boldsymbol{q}) \equiv \frac{1}{\sqrt{2}}\sum_k \tau^{(1),\mathrm{NN}}_{\nu,k}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} = \frac{1}{\sqrt{2}}\sum_{lm}(-1)^l Y^*_{lm}(\hat{\boldsymbol{q}})\rho^{\mathrm{NN}}_{l0lm,1\nu}(q) \tag{12.9}$$

and define its response functions as

$$R_{\nu,\mathrm{S}}(q,\omega) \equiv \sum_n |\langle\Psi^n_{\mathrm{X}}|\left(O_{\nu,\mathrm{S}}(\boldsymbol{q})\right)^\dagger|\Psi^0_{\mathrm{A}}\rangle|^2\delta(\omega-\omega_n) \tag{12.10}$$

Their angular momentaum representation is given by

$$
\begin{aligned}
R_{\nu,\mathrm{S}}(q,\omega) &= \frac{1}{2}\sum_{n'}\sum_{lm} Y^*_{lm}(\hat{\boldsymbol{q}})Y_{lm}(\hat{\boldsymbol{q}})|\langle\Psi^{n'lm}_{\mathrm{X}}|\left(\rho^{\mathrm{NN}}_{l0lm,1\nu}(q)\right)^\dagger|\Psi^0_{\mathrm{A}}\rangle|^2\delta(\omega-\omega_n)\\
&= \frac{1}{8\pi}\sum_l (2l+1)R^{[\mathrm{NN}]}_{ll0l\nu}(q,q;\omega) \qquad (12.11)
\end{aligned}
$$

where we introduced the notations for $R^{[\mathrm{NN}]}$ corresponding to eqs. (9.33) for $\Pi^{[\mathrm{NN}]}$, and used the relation

$$
\sum_m Y^*_{lm}(\hat{\boldsymbol{q}})Y_{lm}(\hat{\boldsymbol{q}}) = \frac{2l+1}{4\pi} \qquad (12.12)
$$

12.2.2 Isovector spin-vector modes

We introduce the isovector spin-vector operators

$$
O^{\mathrm{N}}_{\nu,\mu}(\boldsymbol{q}) \equiv \frac{1}{\sqrt{2}}\sum_k \tau^{(1),\mathrm{NN}}_{\nu,k}\,\sigma^{(1),\mathrm{NN}}_{\mu,k}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} = \frac{1}{\sqrt{2}}\rho^{11,\mathrm{NN}}_{\nu,\mu}(\boldsymbol{q}) \qquad (12.13)
$$

$$
\begin{aligned}
O^{\Delta}_{\nu,\mu}(\boldsymbol{q}) &\equiv \frac{1}{\sqrt{2}}\sum_k \left\{\tau^{(1),\Delta\mathrm{N}}_{\nu,k}\,\sigma^{(1),\Delta\mathrm{N}}_{\mu,k} + \tau^{(1),\mathrm{N}\Delta}_{\nu,k}\,\sigma^{(1),\mathrm{N}\Delta}_{\mu,k}\right\}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}\\
&= \frac{1}{\sqrt{2}}\left\{\rho^{11,\Delta\mathrm{N}}_{\nu,\mu}(\boldsymbol{q}) + \rho^{11,\mathrm{N}\Delta}_{\nu,\mu}(\boldsymbol{q})\right\} \qquad (12.14)
\end{aligned}
$$

and define the isovector spin-vector response function as

$$
R^{ab}_{\nu,\mathrm{V}}(q,\omega) \equiv \sum_n\sum_\mu \langle\Psi^0_{\mathrm{A}}|O^a_{\nu,\mu}(\boldsymbol{q})|\Psi^n_{\mathrm{A}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[O^b_{\nu,\mu}(\boldsymbol{q})\right]^\dagger|\Psi^0_{\mathrm{X}}\rangle\delta(\omega-\omega_n) \qquad (12.15)
$$

where $a,b = \mathrm{N}$ or Δ. They can be rewritten as

$$
\begin{aligned}
R^{\mathrm{NN}}_{\nu,\mathrm{V}}(q,\omega) &= \frac{1}{2}\sum_{JM}\sum_\mu\sum_{l'm'}\sum_{lm}(-1)^{l'+l}Y^*_{l'm'}(\hat{\boldsymbol{q}})(l'm'1\mu|JM)Y_{lm}(\hat{\boldsymbol{q}})(lm1\mu|JM)\\
&\times\sum_{n'}\langle\Psi^0_{\mathrm{A}}|\rho^{\mathrm{NN}}_{l'1JM,1\nu}(q)|\Psi^{n'JM}_{\mathrm{X}}\rangle\langle\Psi^{n'JM}_{\mathrm{X}}|\left[\rho^{\mathrm{NN}}_{l1JM,1\nu}(q)\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n)\\
&= \frac{1}{2}\sum_{JM}\sum_\mu\sum_{l'm'}\sum_{lm}\frac{2J+1}{\sqrt{(2l'+1)(2l+1)}}(J-M1\mu|l'-m')(J-M1\mu|l-m)\\
&\times\; Y^*_{l'm'}(\hat{\boldsymbol{q}})Y_{lm}(\hat{\boldsymbol{q}})R^{\mathrm{NNNN},t'=t=1}_{ls'=1ls=l,J\nu}(q,q;\omega) = \frac{1}{8\pi}\sum_{Jl}(2J+1)R^{[\mathrm{NN}]}_{ll1J\nu}(q,q;\omega) \qquad (12.16)
\end{aligned}
$$

where we used eq. (12.12).

In the same way, we can get

$$
\begin{aligned}
R^{\Delta\mathrm{N}}_{\nu,\mathrm{V}}(q,\omega) &= \frac{1}{2}\sum_{JM}\sum_\mu\sum_{l'm'}\sum_{lm}\sum_{n'}(-1)^{l'+l}Y^*_{l'm'}(\hat{\boldsymbol{q}})(l'm'1\mu|JM)Y_{lm}(\hat{\boldsymbol{q}})(lm1\mu|JM)\\
&\times\;\langle\Psi^0_{\mathrm{A}}|\rho^{\Delta\mathrm{N}}_{l'1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l'1JM,1\nu}(q)|\Psi^{n'JM}_{\mathrm{X}}\rangle\\
&\times\;\langle\Psi^{n'JM}_{\mathrm{X}}|\left[\rho^{\mathrm{NN}}_{l1JM,1\nu}(q)\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n)\\
&= \frac{1}{8\pi}\sum_{Jl}(2J+1)\left\{R^{\Delta\mathrm{NNN},11}_{l1l1J\nu}(q,q;\omega)+R^{\mathrm{N}\Delta\mathrm{NN},11}_{l1l1J\nu}(q,q;\omega)\right\}\\
&= \frac{1}{8\pi}\sum_{Jl}(2J+1)R^{[\Delta\mathrm{N}]}_{ll1J\nu}(q,q;\omega) \qquad (12.17)
\end{aligned}
$$

and similarly

$$R^{\mathrm{N}\Delta}_{\nu,\mathrm{V}}(q,\omega) = \frac{1}{8\pi}\sum_{Jl}(2J+1)R^{[\mathrm{N}\Delta]}_{ll1J\nu}(q,q;\omega) \tag{12.18}$$

We can also derive

$$\begin{aligned}
& R^{\Delta\Delta}_{\nu,\mathrm{V}}(q,\omega) \\
= \quad & \frac{1}{2}\sum_{JM}\sum_{\mu}\sum_{l'm'}\sum_{lm}\sum_{n'}(-1)^{l'+l}Y^{*}_{l'm'}(\hat{\boldsymbol{q}})(l'm'1\mu|JM)Y_{lm}(\hat{\boldsymbol{q}})(lm1\mu|JM)\delta(\omega-\omega_n) \\
\times \quad & \langle\Psi^0_{\mathrm{A}}|\rho^{\Delta\mathrm{N}}_{l'1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l'1JM,1\nu}(q)|\Psi^{n'JM}_{\mathrm{X}}\rangle\langle\Psi^{n'JM}_{\mathrm{X}}|\left[\rho^{\Delta\mathrm{N}}_{l1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l1JM,1\nu}(q)\right]^{\dagger}|\Psi^0_{\mathrm{A}}\rangle \\
= \quad & \frac{1}{8\pi}\sum_{Jl}(2J+1)\Big\{R^{\Delta\mathrm{N}\Delta\mathrm{N},11}_{l1l1J,\nu}(q,q;\omega)+R^{\mathrm{N}\Delta\Delta\mathrm{N},11}_{l1l1J,\nu}(q,q;\omega) \\
+ \quad & R^{\Delta\mathrm{N}\mathrm{N}\Delta,11}_{l1l1J,\nu}(q,q;\omega)+R^{\mathrm{N}\Delta\mathrm{N}\Delta,11}_{l1l1J,\nu}(q,q;\omega)\Big\} = \frac{1}{8\pi}\sum_{Jl}(2J+1)R^{[\Delta\Delta]}_{ll1J,\nu}(q,q;\omega)
\end{aligned} \tag{12.19}$$

In the above we introduced the notations for $R^{[\Delta\mathrm{N}]}, R^{[\mathrm{N}\Delta]}$ and $R^{[\Delta\Delta]}$ corresponding to $\Pi^{[\Delta\mathrm{N}]}, \Pi^{[\mathrm{N}\Delta]}$ and $\Pi^{[\Delta\Delta]}$ of eqs. (9.34), (9.35) and (9.36).

In summary we can write

$$R^{ab}_{\nu,\mathrm{V}}(q,\omega) = \frac{1}{8\pi}\sum_{Jl}(2J+1)R^{[ab]}_{ll1J\nu}(q,q;\omega), \qquad (a,b=\mathrm{N}\text{ or }\Delta) \tag{12.20}$$

12.2.3 Isovector spin-longitudinal mode

We consider the isovector spin-longitudinal operators

$$O^{\mathrm{N}}_{\nu,\mathrm{L}}(\boldsymbol{q}) \equiv \frac{-1}{\sqrt{2}}\sum_{k}\tau^{(1),\mathrm{NN}}_{\nu,k}(\boldsymbol{\sigma}^{(1),\mathrm{NN}}_{k}\cdot\hat{\boldsymbol{q}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} \tag{12.21}$$

$$O^{\Delta}_{\nu,\mathrm{L}}(\boldsymbol{q}) \equiv \frac{-1}{\sqrt{2}}\sum_{k}\left\{\tau^{(1),\Delta\mathrm{N}}_{\nu,k}(\boldsymbol{\sigma}^{(1),\Delta\mathrm{N}}_{k}\cdot\hat{\boldsymbol{q}})+\tau^{(1),\mathrm{N}\Delta}_{\nu,k}(\boldsymbol{\sigma}^{(1),\mathrm{N}\Delta}_{k}\cdot\hat{\boldsymbol{q}})\right\}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} \tag{12.22}$$

which can be written in the angular momentum representation as

$$\begin{aligned}
O^{\mathrm{N}}_{\nu,\mathrm{L}}(\boldsymbol{q}) &= \frac{4\pi}{\sqrt{2}}\sum_{lJM}a_{Jl}(-1)^{J}Y^{*}_{JM}(\hat{\boldsymbol{q}})\sum_{k}\tau^{(1),\mathrm{NN}}_{\nu,k}j_l(qr_k)\left[\mathrm{i}^{l}Y_l(\hat{\boldsymbol{r}}_k)\times\sigma^{(1),\mathrm{NN}}_{k}\right]^{J}_{M} \\
&= \frac{1}{\sqrt{2}}\sum_{lJM}a_{Jl}(-1)^{J}Y^{*}_{JM}(\hat{\boldsymbol{q}})\rho^{\mathrm{NN}}_{l1JM,1\nu}(q)
\end{aligned} \tag{12.23}$$

and similarly

$$O^{\Delta}_{\nu,\mathrm{L}}(\boldsymbol{q}) = \frac{1}{\sqrt{2}}\sum_{lJM}a_{Jl}(-1)^{J}Y^{*}_{JM}(\hat{\boldsymbol{q}})\left(\rho^{\Delta\mathrm{N}}_{l1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l1JM,1\nu}(q)\right) \tag{12.24}$$

where eqs.(6.19) and (6.20) are used.

We then define the isovector spin-longitudinal response function as

$$R^{ab}_{\nu,\mathrm{L}}(q,\omega) \equiv \sum_{n}\langle\Psi^0_{\mathrm{A}}|O^{a}_{\nu,\mathrm{L}}(\boldsymbol{q})|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[O^{b}_{\nu,\mathrm{L}}(\boldsymbol{q})\right]^{\dagger}|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \tag{12.25}$$

They can be written as

$$\begin{aligned}
R^{\mathrm{NN}}_{\nu,\mathrm{L}}(q,\omega) &= \frac{1}{2}\sum_{n}\sum_{JM}\sum_{l'l} a_{Jl'}a_{Jl}Y^*_{JM}(\hat{\boldsymbol{q}})Y_{JM}(\hat{\boldsymbol{q}}) \\
&\times \langle\Psi^0_{\mathrm{A}}|\rho^{\mathrm{NN}}_{l'1JM,1\nu}(q)|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[\rho^{\mathrm{NN}}_{l1JM,1\nu}(q)\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \\
&= \frac{1}{8\pi}\sum_{J}\sum_{l'l}(2J+1)a_{Jl'}a_{Jl}R^{\mathrm{NNNN}}_{l'1l1J,\nu}(q,q;\omega) \\
&= \frac{1}{8\pi}\sum_{J}\sum_{l'l}(2J+1)a_{Jl'}a_{Jl}R^{[\mathrm{NN}]}_{l'l1J\nu}(q,q;\omega)
\end{aligned} \tag{12.26}$$

and similarly

$$\begin{aligned}
R^{\Delta\mathrm{N}}_{\nu,\mathrm{L}}(q,\omega) &= \frac{1}{2}\sum_{n}\sum_{JM}\sum_{l'l} a_{Jl'}a_{Jl}Y^*_{JM}(\hat{\boldsymbol{q}})Y_{JM}(\hat{\boldsymbol{q}})\delta(\omega-\omega_n) \\
&\times \langle\Psi^0_{\mathrm{A}}|\rho^{\Delta\mathrm{N}}_{l'1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l'1JM,1\nu}(q)|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[\rho^{\mathrm{NN}}_{l1JM,1\nu}(q)\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle \\
&= \frac{1}{8\pi}\sum_{J}\sum_{l'l}(2J+1)a_{Jl'}a_{Jl}R^{[\Delta\mathrm{N}]}_{l'l1J\nu}(q,q;\omega)
\end{aligned} \tag{12.27}$$

$$\begin{aligned}
&R^{\Delta\Delta}_{\nu,\mathrm{L}}(q,\omega) \\
&= \frac{1}{2}\sum_{n}\sum_{JM}\sum_{l'l} a_{Jl'}a_{Jl}Y^*_{JM}(\hat{\boldsymbol{q}})Y^*_{JM}(\hat{\boldsymbol{q}})\delta(\omega-\omega_n) \\
&\times \langle\Psi^0_{\mathrm{A}}|\rho^{\Delta\mathrm{N}}_{l'1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l'1JM,1\nu}(q)|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[\rho^{\Delta\mathrm{N}}_{l1JM,1\nu}(q)+\rho^{\mathrm{N}\Delta}_{l1JM,1\nu}(q)\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle \\
&= \frac{1}{8\pi}\sum_{J}\sum_{l'l}(2J+1)a_{Jl'}a_{Jl}R^{[\Delta\Delta]}_{l'l1J\nu}(q,q;\omega)
\end{aligned} \tag{12.28}$$

In summary we can write

$$R^{ab}_{\nu,\mathrm{L}}(q,\omega) = \frac{1}{8\pi}\sum_{J}\sum_{l'l}(2J+1)a_{Jl'}a_{Jl}R^{[ab]}_{l'l1J\nu}(q,q;\omega) \qquad (a,b=\mathrm{N}\text{ or }\Delta) \tag{12.29}$$

12.2.4 Isovector spin-transverse modes

Introducing the isovector spin-transverse operators

$$O^{\mathrm{N}}_{\nu,\mu,\mathrm{T}}(\boldsymbol{q}) \equiv -\frac{1}{2}\sum_{k}\tau^{(1),\mathrm{N}}_{\nu,k}\left[\boldsymbol{\sigma}^{(1),\mathrm{N}}_{k}\times\hat{\boldsymbol{q}}\right]_\mu \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} \tag{12.30}$$

$$O^{\Delta}_{\nu,\mu,\mathrm{T}}(\boldsymbol{q}) \equiv -\frac{1}{2}\sum_{k}\left\{\tau^{(1),\Delta\mathrm{N}}_{\nu,k}\left[\boldsymbol{\sigma}^{(1),\Delta\mathrm{N}}_{k}\times\hat{\boldsymbol{q}}\right]_\mu+\tau^{(1),\mathrm{N}\Delta}_{\nu,k}\left[\boldsymbol{\sigma}^{(1),\mathrm{N}\Delta}_{k}\times\hat{\boldsymbol{q}}\right]_\mu\right\}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} \tag{12.31}$$

we define the isovector spin-transverse response functions as

$$R^{ab}_{\nu,\mathrm{T}}(q,\omega) \equiv \sum_{\mu}\sum_{n}\langle\Psi^0_{\mathrm{A}}|O^{a}_{\nu\mu,\mathrm{T}}(\boldsymbol{q})|\Psi^n_{\mathrm{X}}\rangle\langle\Psi^n_{\mathrm{X}}|\left[O^{b}_{\nu\mu,\mathrm{T}}(\boldsymbol{q})\right]^\dagger|\Psi^0_{\mathrm{A}}\rangle\delta(\omega-\omega_n) \tag{12.32}$$

Noting the relation $[\boldsymbol{A}\times\hat{\boldsymbol{q}}]\cdot[\boldsymbol{B}\times\hat{\boldsymbol{q}}] = (\boldsymbol{A}\cdot\boldsymbol{B})-(\boldsymbol{A}\cdot\hat{\boldsymbol{q}})(\boldsymbol{B}\cdot\hat{\boldsymbol{q}})$, we get

$$R^{ab}_{\nu,\mathrm{T}}(q,\omega) = \frac{1}{2}\left\{R^{ab}_{\nu,\mathrm{V}}(q,\omega) - R^{ab}_{\nu\mathrm{L}}(q,\omega).\right\} \tag{12.33}$$

12.2.5 Full response functions in N+Δ space

To show the combined responses from N and Δ, we conventionally define the full isovector spin operators

$$
\begin{aligned}
& \boldsymbol{\tau\sigma} + \frac{f_{\pi \mathrm{N}\Delta}}{f_{\pi \mathrm{NN}}} \left\{ \boldsymbol{TS} + (\boldsymbol{TS})^{\dagger} \right\} \\
= \; & \boldsymbol{\tau}^{(1),\mathrm{NN}} \boldsymbol{\sigma}^{(1),\mathrm{NN}} + \frac{f_{\pi \mathrm{N}\Delta}}{f_{\pi \mathrm{NN}}} \left\{ \boldsymbol{\tau}^{(1),\Delta\mathrm{N}} \boldsymbol{\sigma}^{(1),\Delta\mathrm{N}} + \boldsymbol{\tau}^{(1),\mathrm{N}\Delta} \boldsymbol{\sigma}^{(1),\mathrm{N}\Delta} \right\}
\end{aligned} \tag{12.34}
$$

and present the full isovector spin response functions defined as

$$
\begin{aligned}
& R_{\nu,\mathrm{X}}(q,\omega) \\
& = R^{\mathrm{NN}}_{\nu,\mathrm{X}}(q,\omega) + \frac{f_{\pi \mathrm{N}\Delta}}{f_{\pi \mathrm{NN}}} \left(R^{\mathrm{N}\Delta}_{\nu,\mathrm{X}}(q,\omega) + R^{\Delta\mathrm{N}}_{\nu,\mathrm{X}}(q,\omega) \right) + \left(\frac{f_{\pi \mathrm{N}\Delta}}{f_{\pi \mathrm{NN}}} \right)^2 R^{\Delta\Delta}_{\nu,\mathrm{X}}(q,\omega)
\end{aligned} \tag{12.35}
$$

where X = V, L or T.

References

[1] K. Kawahigashi, K. Nishida, A. Itabashi, and M. Ichimura, Phys. Rev. **C63**(2001), 044609.

[2] M. Ichimura, H. Sakai , and T. Wakasa, Prog. Part. Nucl. Phys. **56**(2006), 446.

[3] K. Nishida, and M. Ichimura, Phys. Rev. **C51**(1995), 269.

[4] M. Ichimura, and K. Kawahigashi, Phys. Rev. **C45**(1992), 1822; Errata. Phys. Rev. **C46**(1992)

[5] Y. Nakaoka and M. Ichimura, Prog. Theoret. Phys.. **102**(1999), 599

[6] R. Hagedorn, "Relativistic Kinematics", W.A. Benjamin, Inc. (1963)

[7] W. G. Love, and M. A. Franey, Phys. Rev. **C24**(1981), 1073

[8] M. A. Franey and W. G. Love, Phys. Rev. **C31**(1985), 488

[9] A.K.Kerman, H.McManus and R.M.Thaler, Ann. of Phys. **8** (1959) 551

[10] T. Izumoto and A. Mori, Phys. Lett. 82B(1979)163

[11] G. R. Satchler, Direct Nuclear Reactions, Clarendon, Oxford, 1983

[12] R. D Smith and J. Wambach, Phys. Rev. C **38** (1988) 100

[13] A. Bohr and B. Mottelson, Nuclear Structure vol.1, W.A. Benjamin, Inc. , 1989