Heisenberg uncertainty relation revisited†

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Kennard and Robertson formulated the uncertainty relation which appears in any textbook on quantum mechanics

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|\langle[A,B]\rangle|. \tag{1}$$

Another important development in the history of uncertainty relations is the analysis of Arthurs and Kelly[1]. They introduce the measuring apparatus M for A, and N for B, respectively, with $[M,N]=0$. The notion of unbiased measurement is important in their analysis, which is defined by

$$\langle M^{out}\rangle = \langle A\rangle \tag{2}$$

for *any* state of the system ψ in the total Hilbert space of the system and apparatus $|\psi\rangle\otimes|\xi\rangle$ in the Heisenberg picture. Here variables M and N (and also A and B) stand for the variables before the measurement, and the variable $M^{out}=U^{\dagger}MU$ stands for the apparatus M after measurement.

Traditionally, it has been common to take the relation[2]

$$\sigma(M^{out}-A)\sigma(B^{out}-B) \geq \frac{1}{2}|\langle[A,B]\rangle| \tag{3}$$

as the naive Heisenberg error-disturbance relation; we use the adjective "naive" since no reliable derivation of this relation is known. An elegant experiment of spin measurement by J. Erhart et al.[3], invalidated the naive Heisenberg-type error-disturbance relation, which initiated the recent activities on uncertainty relations.

It is shown that all the uncertainty relations are derived from suitably defined Robertson's relation[4]. We start with Robertson's relation

$$\sigma(M^{out}-A)\sigma(B^{out}-B) \geq \frac{1}{2}|\langle[M^{out}-A, B^{out}-B]\rangle| \tag{4}$$

and use the triangle inequality

$$\begin{aligned}&\sigma(M^{out}-A)\sigma(B^{out}-B)\\ &\geq \frac{1}{2}\{|\langle[A,B]\rangle| - |\langle[A,B^{out}-B]\rangle|\\ &\quad -|\langle[M^{out}-A,B]\rangle|\},\end{aligned} \tag{5}$$

where we used $[M^{out},B^{out}]=[M,B]=0$. Using the variations of Robertson's relation, we obtain[2]

$$\begin{aligned}&\sigma(M^{out}-A)\sigma(B^{out}-B)+\sigma(M^{out}-A)\sigma(B)\\ &+\sigma(A)\sigma(B^{out}-B) \geq \frac{1}{2}|\langle[A,B]\rangle|,\end{aligned} \tag{6}$$

and[5]

$$\begin{aligned}&\{\sigma(M^{out}-A)+\sigma(A)\}\{\sigma(B^{out}-B)+\sigma(B)\}\\ &\geq |\langle[A,B]\rangle|.\end{aligned} \tag{7}$$

We thus conclude that all the known universally valid relations are the secondary consequences of Robertson's relation. Also, the saturation of Robertson's relation is a *necessary condition* of the saturation of universally valid uncertainty relations. If one assumes the unbiased measurement and disturbance, one obtains (3).

By assuming unbiased joint measurements, we conclude[6]

$$\langle[A,B]\rangle = \langle[M^{out},N^{out}]\rangle = 0 \tag{8}$$

which is a contradiction since $\langle[A,B]\rangle \neq 0$ in general. Similarly, one concludes[6]

$$\langle[A,B]\rangle = \langle[M^{out},B^{out}]\rangle = 0 \tag{9}$$

if one assumes the precise measurement of A and the unbiased disturbance of B which implies $\langle B^{out}-B\rangle = 0$ for all ψ. Here $B^{out}=U^{\dagger}(B\otimes 1)U$ stands for the variable B after the *measurement* of A. Note that $[M^{out},B^{out}]=[M,B]=0$.

We interpret the algebraic inconsistency (9) as an indication of the failure of the assumption of unbiased disturbance of B for the precise projective measurement of A, if all the operators involved are *well-defined*. Thus the naive relation (3) fails. On the other hand, the Heisenberg error-error relation

$$\epsilon(M^{out}-A)\epsilon(N^{out}-B) \geq \frac{1}{2}|\langle[A,B]\rangle| \tag{10}$$

and the Arthurs-Kelly relation

$$\sigma(M^{out})\sigma(N^{out}) \geq |\langle[A,B]\rangle| \tag{11}$$

are expected to be valid as conditionally valid uncertainty relations. In this case the apparatus variable N^{out} becomes *singular* for the precise measurement of A, namely, $M^{out}-A \to 0$ if the unbiasedness condition $\langle N^{out}-B\rangle = 0$ is imposed.

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