Note on intersecting branes in topological strings

T. Kimura[*1]

The intersection of branes is an important object in string theory, in order to study non-perturbative aspects of branes, and also its applicatioins to quantum field theory. In this report we investigate some aspects of the intersecting branes in topological string theory, especially through its matrix model description.

We consider the topological B-model on the Calabi-Yau threefold $uv - H(p,x) = 0$ with $H(p,x) = p^2 - W'(x)^2 - f(x)$. This geometry realizes at the large N limit of the matrix model with the potential function $W(x)$. There are seemingly two kinds of non-compact branes in the topological B-model, which correspond to the characteristic polynomial and the external source in the matrix model.[1)] They play a role of the creation operator of branes for x and p coordinates, respectively. By considering both kinds of the branes simultaneously, we can discuss intersection of branes in the B-model. The corresponding matrix model partition function $\Psi_{N,M}(\{a_j\};\{\lambda_\alpha\})$ is given by

$$\int_{N\times N} dX\, e^{-\frac{1}{g_s}\mathrm{Tr}W(X)+\mathrm{Tr}AX} \prod_{\alpha=1}^{M} \det(\lambda_\alpha - X)\,. \qquad (1)$$

This is the M-point function of characteristic polynomials in $N\times N$ Hermitian matrix model with external source A. In order to evaluate the partition function (1), we first rewrite it only in terms of eigenvalues by integrating out the angular part of the matrix X. Then, after some calculations, we obtain the determinantal expression of the partition function

$$\frac{1}{\Delta(a)\Delta(\lambda)} \det \begin{pmatrix} Q_{j-1}(a_k) & Q_{N+\alpha-1}(a_k) \\ P_{j-1}(\lambda_\beta) & P_{N+\alpha-1}(\lambda_\beta) \end{pmatrix}, \qquad (2)$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, and $P_k(x) = x^k + \cdots$ is arbitrary k-th monic polynomial. The function $Q_k(a)$ is the Fourier (Laplace) transform of $P_k(x)\, e^{-\frac{1}{g_s}W(x)}$. Therefore, from the expression (2), we can see an explicit duality between $\vec{a}$ and $\vec{\lambda}$ through the Fourier transformation. In terms of the topological strings, this duality reflects the symplectic invariance of the canonical pair (p,x) in the B-model, which is also seen as the open/closed string duality. We also note that this kind of symplectic invariance appears quite generally in the topological expansion of the spectral curve.

If we apply the Gaussian potential, two functions $P_k(x)$ and $Q_k(x)$ are essentially equivalent, since it is self-dual against the Fourier transformation. In this case we can rewrite the partition function in terms of $\mathrm{U}(N)\times\mathrm{U}(M)$ bifundamental chiral fermions, which are seen as effective degrees of freedom on the intersecting branes. The corresponding effective action is given by

$$S_{\text{eff}} = \frac{g_s}{2}\psi_i^\alpha \bar{\psi}_j^\alpha \psi_j^\beta \bar{\psi}_i^\beta + \mathrm{Tr}A\psi^\alpha\bar{\psi}^\alpha - \mathrm{Tr}\Lambda\psi_j\bar{\psi}_j\,, \qquad (3)$$

In this expression the duality between A and Λ is manifest. In this action the full symmetry of $\mathrm{U}(N)\times\mathrm{U}(M)$ is partially broken due to the source terms.

Let us then comment on the integrability of the brane intersection partition function (2). This kind of determinantal formula generically plays a role as the τ-function,[2)] and satisfies the Toda lattice equation by taking the equal parameter limit. To show that, we now consider the equal position limit of (2) as $a_j \to a$ and $\lambda_\alpha \to \lambda$. Then we have

$$\prod_{j=0}^{N-1}\frac{1}{j!}\prod_{\alpha=0}^{M-1}\frac{1}{\alpha!}\det\begin{pmatrix} Q_{j-1}^{(k-1)}(a) & Q_{N+\alpha-1}^{(k-1)}(a) \\ P_{j-1}^{(\beta-1)}(\lambda) & P_{N+\alpha-1}^{(\beta-1)}(\lambda)\end{pmatrix}. \qquad (4)$$

This is just a hybritized version of Wronskian. If we apply the simplest choice $P_k(x) = x^k$, $Q_k(a)$ is just given as k-th derivative of the generalized Airy function

$$Q_k(a) = \left(\frac{d}{da}\right)^k \int dx\, e^{-\frac{1}{g_s}W(x)+ax}\,. \qquad (5)$$

Applying the Jacobi identity for determinants to the expression (4), we obtain the following 3-term reccurence relations

$$\frac{\tilde{\Psi}_{N+1,M}\cdot\tilde{\Psi}_{N-1,M}}{\left(\tilde{\Psi}_{N,M}\right)^2} = \frac{N}{M}\frac{\partial^2}{\partial a^2}\log\tilde{\Psi}_{N,M}\,, \qquad (6)$$

$$\frac{\tilde{\Psi}_{N,M+1}\cdot\tilde{\Psi}_{N,M-1}}{\left(\tilde{\Psi}_{N,M}\right)^2} = \frac{\partial^2}{\partial a\partial\lambda}\log\tilde{\Psi}_{N,M}\,. \qquad (7)$$

where we have rescaled the partition function as $\tilde{\Psi}_{N,M}(a,\lambda) = e^{-\lambda}\Psi_{N,M}(a,\lambda)$. The equations (6) and (7) are the Toda lattice equations in one and two dimensions. This means that the brane intersection partition function is the τ-function for the Toda lattice hierarchy in both senses. We can also introduce infinitely many "time" variables for this τ-function in the Miwa coordinate

$$t_n = \frac{1}{n}\mathrm{Tr}\,A^{-n}\,, \quad \tilde{t}_n = \frac{1}{n}\mathrm{Tr}\,\Lambda^{-n}\,. \qquad (8)$$

If we take the large N limit of the matrix model, which corresponds to the continuum limit of the Toda lattice equation, we obtain the KdV/KP integrable equations.

References

1) M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, C. Vafa, Commun. Math. Phys. **261** (2006) 451–516.
2) A. Morozov, Phys. Usp. **37** (1994) 1–55.

*1 RIKEN Nishina Center