Duality and integrability of supermatrix model with external source†

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In quantum field theory, in order to compute correlation functions, it is convenient to introduce the generating function by adding an extra source term. Such a generating function is defined in the sense of path integral, and thus it is quite difficult to compute in general. However, in the matrix model, just a zero dimensional theory, a number of methods for computation are established, which are also applicable to the model with the external source. In this report we generalize the duality of the matrix model with the external source with a characteristic polynomial, which was originally found in the Gaussian matrix model,[1)] to the supermatrix model with an arbitrary matrix potential.

The correlation function of the characteristic polynomial in the supermatrix model, which we study here, is given by

$$\Psi_{N,M;p,q}\left(\{a_i\}_{i=1}^N,\{b_j\}_{j=1}^M;\{\lambda_\alpha\}_{\alpha=1}^p,\{\mu_\beta\}_{\beta=1}^q\right) = \int dZ\, e^{-\frac{1}{\hbar}\operatorname{Str}W(Z)+\operatorname{Str}ZC}\,\frac{\prod_{\alpha=1}^p \operatorname{Sdet}(\lambda_\alpha - Z)}{\prod_{\beta=1}^q \operatorname{Sdet}(\mu_\beta - Z)} \quad (1)$$

where Z is a size $N+M$ Hermitian supermatrix, and the external source is $C=\operatorname{diag}(a_1,\dots,a_N,b_1,\dots,b_M)$. This formula includes several useful situations, e.g., the ordinary characteristic polynomial average ($M=q=0$), the average of inverses ($M=p=0$), and the ratio average ($M=0$). Therefore it provides a master formula for the characteristic polynomial average in various matrix models.

The matrix measure in the integral is invariant under the supergroup transformation, $dZ=d(UZU^{-1})$ with $U\in \mathrm{U}(N|M)$, which is expressed in terms of eigenvalues, $dZ=\Delta_{N,M}(x;y)^2\, d^N x\, d^M y\, dU$. Here the Jacobian is given by the Cauchy determinant,

$$\Delta_{N,M}(x;y) = \frac{\prod_{i<j}^N (x_i-x_j)\prod_{i<j}^M (y_i-y_j)}{\prod_{i,j}^{N,M}(x_i-y_j)}\,. \quad (2)$$

Then, to compute the intgral, we now introduce the Harish-Chandra–Itzykson–Zuber formula for the supergroup $\mathrm{U}(N|M)$[2–4)]

$$\int_{\mathrm{U}(N|M)} dU\, e^{\operatorname{Str}ZUCU^{-1}} = \frac{\det e^{x_i a_i}\ \det e^{-y_i b_j}}{\Delta_{N,M}(x;y)\Delta_{N,M}(a;b)}\,. \quad (3)$$

Applying this formula, we obtain the following expression for the matrix integral in terms of eigenvalues

$$\Psi_{N,M;p,q} = \int \prod_{i,j}^{N,M} dx_i\, dy_j\, e^{-\frac{1}{\hbar}W(x_i)+\frac{1}{\hbar}W(y_j)+x_i a_i - y_j b_j} \times \frac{\Delta_{N+p,M+q}(x,\lambda;y,\mu)}{\Delta_{N,M}(a;b)\Delta_{p,q}(\lambda;\mu)}\,. \quad (4)$$

Since the Cauchy determinant can be written as a determinant

$$\Delta_{N,M}(x;y) = \det\begin{pmatrix} x_i^{k-1} \\ (x_i-y_j)^{-1}\end{pmatrix} \quad (5)$$

with $i=1,\dots,N$, $j=1,\dots,M$, $k=1,\dots,N-M$, if $N\geq M$, we obtain the determinanal formula for the characteristic polynomial average

$$\Psi_{N,M;p,q}\left(\{a_i\}_{i=1}^N,\{b_j\}_{j=1}^M;\{\lambda_\alpha\}_{\alpha=1}^p,\{\mu_\beta\}_{\beta=1}^q\right) = \frac{1}{\Delta_{N,M}(a;b)\Delta_{p,q}(\lambda;\mu)}\det\begin{pmatrix} Q_{k-1}(a_i) & P_{k-1}(\lambda_\alpha) \\ R(a_i;b_j) & S_{\mathrm{R}}(\lambda_\alpha;b_j) \\ S_{\mathrm{L}}(a_i;\mu_\beta) & \tilde{R}_{\lambda_\alpha;\mu_\beta}\end{pmatrix}, \quad (6)$$

where we have introduced auxiliary functions:

$$P_{i-1}(x) = x^{i-1}\,,\quad \tilde{R}(x;y)=\frac{1}{x-y}\,, \quad (7)$$

$$Q_{i-1}(a) = \int dx\, P_{i-1}(x)\, e^{-\frac{1}{\hbar}W(x)+xa}\,, \quad (8)$$

$$R(a;b) = \int dxdy\, \tilde{R}(x;y)\, e^{-\frac{1}{\hbar}(W(x)-W(y))+xa-yb}\,, \quad (9)$$

$$S_{\mathrm{L}}(a;\mu) = \int dx \frac{1}{x-\mu} e^{-\frac{1}{\hbar}W(x)+xa}\,, \quad (10)$$

$$S_{\mathrm{L}}(\lambda;b) = \int dy \frac{1}{\lambda-y} e^{\frac{1}{\hbar}W(y)-yb}\,. \quad (11)$$

The formula (6) actually shows a duality between the external source and the characteristic polynomial, which is just given by Laplace (Fourier) transforms,

$$\Psi_{N,M;p,q}\left(\{a_i\}_{i=1}^N,\{b_j\}_{j=1}^M;\{\lambda_\alpha\}_{\alpha=1}^p,\{\mu_\beta\}_{\beta=1}^q\right) \overset{\text{F.T.}}{=} \Psi_{p,q;N,M}\left(\{\lambda_\alpha\}_{\alpha=1}^p,\{\mu_\beta\}_{\beta=1}^q;\{a_i\}_{i=1}^N,\{b_j\}_{j=1}^M\right). \quad (12)$$

This is because the auxiliary functions transform to each other through the Fourier transformation: $P_{i-1}(x) \leftrightarrow Q_{i-1}(a)$, $R(x;y)\leftrightarrow \tilde{R}(a;b)$, $S_{\mathrm{L}}(\lambda;b)\leftrightarrow S_{\mathrm{R}}(a;\mu)$.

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