Separability criteria with angular and Hilbert-space averages†

K. Fujikawa,[*1] C.H. Oh,[*2] K. Umetsu,[*3] and Sixia Yu[2]

To characterize entanglement, the notions of separability and inseparability, which are the characteristic properties of state vectors in quantum mechanics, are commonly used. On the other hand, the notion of local realism based on local non-contextual hidden-variable models is also used to test properties such as locality and reduction.[1)] Local hidden-variable models are generally different from quantum mechanics, and thus local realism tests the deviation from quantum mechanics also.

In the experimental study of local realism, which is commonly tested by the CHSH inequality,[2)] it is customary to first confirm the consistency of the measured basic correlation such as the spin correlation $\langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle$ with quantum mechanics and then test the CHSH inequality. If one confirms the consistency of $\langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle$ for any unit vectors $\mathbf{a}$ and $\mathbf{b}$ with quantum mechanics, one can naturally apply the criterion of quantum mechanical separability to the correlation. In the present study, we discuss this aspect of the separability test in quantum mechanics. It is well known that the Peres criterion of the positivity of a partial transposed density matrix gives a necessary and sufficient condition for the separability of a general density matrix in $d = 2\times 2$[3)] which we study in the present paper. However, it is also well-known from the days of Pauli that the reconstruction of the state vector or density matrix from measured data is generally very involved.[4)] It is thus practically useful to derive simpler criteria that do not require precise state reconstruction. The purpose of the present paper is to derive such separability criteria. We assume the most general separable density matrix, but we use a limited set of two-point correlations and thus obtain only the necessary condition of separability in general. Nevertheless, we illustrate that our criteria are very useful when applied to the past experimental data of Aspect et al. in 1981[5)] and Sakai et al. in 2006.[6)]

To be specific, we study the general separable quantum mechanical states

$$\rho = \sum_k w_k\rho_k \tag{1}$$

in $d = 2\times 2 = 4$; all the states ρ_k are separable pure quantum states.[7)] If the state is a separable pure state $\rho = |\psi\rangle\langle\psi|$, separability in (1) is quantified by the equality, which is necessary and sufficient,

$$G(\mathbf{a},\mathbf{b}) \equiv 4[\langle\psi|P(\mathbf{a})\otimes P(\mathbf{b})|\psi\rangle - \langle\psi|P(\mathbf{a})\otimes\mathbf{1}|\psi\rangle\langle\psi|\mathbf{1}\otimes P(\mathbf{b})|\psi\rangle] = 0 \tag{2}$$

for two arbitrary projection operators $P(\mathbf{a})$ and $P(\mathbf{b})$.

In the case of a general mixed ρ, we derive the useful criteria of separable mixed states, namely

$$\mathrm{Tr}\rho\{\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\} = (1/3)C\cos\varphi \tag{3}$$

for two spin 1/2 systems, and

$$4\mathrm{Tr}\rho\{P(\mathbf{a})\otimes P(\mathbf{b})\} = 1 + (1/2)C\cos 2\varphi \tag{4}$$

for two photon systems with linear polarization projectors $P(\mathbf{a})$ and $P(\mathbf{b})$, respectively. Here, $-1\le C\le 1$. The basic new ingredient in our derivation of criteria (3) and (4) is that we take a geometric *angular average* of unit vectors $\mathbf{a}$ and $\mathbf{b}$ with fixed $\cos\varphi = \mathbf{a}\cdot\mathbf{b}$. It is shown that this angular averaging, which was originally motivated by the specific experiment,[6)] does not add a new burden on the measurements by analyzing the existing experimental data. The difference in the numerical coefficients 1/3 and 1/2 in (3) and (4) arises from the difference in rotational properties; the spinor rotational freedom is 3-dimensional, which agrees with the freedom of the $d = 2$ Hilbert space, while the rotational freedom of the photon is two-dimensional, which differs from the freedom of the $d = 2$ Hilbert space because it is confined in a plane perpendicular to the momentum direction. If one instead takes an average over the states in the $d = 2$ Hilbert space, the formula for the photon is replaced by

$$4\mathrm{Tr}\rho\{P(\mathbf{a})\otimes P(\mathbf{b})\} = 1 + (1/3)C\cos 2\varphi. \tag{5}$$

This difference between the geometrical angular average and the Hilbert space average is interesting, and it has an interesting implication on the separability issue of the entangled Werner state[7)] which accommodates a specific local hidden-variable representation and thus satisfies the CHSH inequality. In the photon case, the Hilbert space average can judge the inseparability of the Werner state but the geometrical angular average cannot, while the geometric average can judge the Werner state in the case of spin 1/2 systems.

References

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*1 RIKEN Nishina Center
*2 Department of Physics, National University of Singapore
*3 College of Science and Technology, Nihon University