Algebra of the QST model

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The standard model (SM) successully describes nature; however it has at least 25 free parameters. Simple formulas with no free parameter for 24 SM parameters have been reported.[1)] Further the quarternion-spin-isospin (QST) model, which is an algebraic model that predicts these formulas,[2)] and its implication to gravity and cosmology[3)] have also been reported. This article describes the algebra of the QST model.

In the QST model, the Planck time $t_{\mathrm{pl}} = 5.3912 \times 10^{-44}$ s is the minimum time period in nature, and it is a fundamental constant in nature, similar to the speed of light in vacuum c and the Planck constant $\hbar$. Since t_{pl} represents the mininum time, the QST model cannot be a continuous theory. It is assumed that nature is not continuous at the fundamental level, and that a contiuous theory such as the SM is a continuous approximation of the QST model.

In a quantum field theory (QFT), physical states are assumed to be represented by a Hilbert space, which is a real or complex vector space. In a QFT, the change of the state vector $|t_i\rangle$ at $t = t_i$ to $|t_f\rangle$ at $t = t_f$is given by the following path integral:

$$|t_f\rangle = \int \mathcal{D}\phi \exp\Big(i \int \mathcal{L}[\phi(x,t)] d^4x\Big)|t_i\rangle,$$

where $\mathcal{L}[\phi(x,t)]$ represents the Lagrangian density of the QFT. For a very short time period δt, this formula can be written as

$$|\vec{x}, t+\delta t\rangle = \Big(1 + i\mathcal{L}[\phi(\vec{x},t)]\delta^3\vec{x}\delta t\Big)|\vec{x},t\rangle.$$

Thus, the change of the physical state is represented by the action $i\delta S = i\mathcal{L}[\phi(\vec{x},t)]\delta^3\vec{x}\delta t$. It is assumed that $U = 1 + i\delta S$ is an unitary operator. This means that U in a QFT represents a reversible operator acting on a vector space that represnts the physical states.

In the QST model, physical actions are represented by algebraic structure called a $\mathbb{Z}$-module in mathematics. In the following, we first explain the mathematical terms necessary to understand the QST model.

A $\mathbb{Z}$-module is a generalization of the notion of vector space in which the field of scalars (*e.g.* the field of real numbers $\mathbb{R}$ for a real vector space) is replaced by the ring of rational integers $\mathbb{Z}$. A $\mathbb{Z}$-module is like a vector space; however the scalar multiplication factor a of a vector v ($v \to av$) is limied to an integer. Any abelian group can be considered as a $\mathbb{Z}$-module. A $\mathbb{Z}$-module M is called finitely generated if there exist a set of a finite number of elements $\{e_1, e_2, \cdots, e_s\}$ in M such that any element m in M can be written as

$$m = n_1e_1 + n_2e_2 + \cdots + n_se_s (n_k \in \mathbb{Z}).$$

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The set $\{e_1, e_2, \cdots, e_s\}$ is called a generating set of M. A generating set is called basis if its all elements are linearly independent of each other. A vector space always has a basis; however a $\mathbb{Z}$ module does not necessarily have a basis. A $\mathbb{Z}$-module that has a basis is called as a free $\mathbb{Z}$-module. Let $\mathbb{Z}^M[S]$ represent a free $\mathbb{Z}$-module with S as the basis. If S is a generating set of M, there exists a mapping function (homomorphism) ϕ_M from $\mathbb{Z}^M[S]$ onto M that preserves the structure of the module.

$$\mathbb{Z}^M[S] \overset{\phi_M}{\to} M \to 0$$

A finitely generated free $\mathbb{Z}$-module M is isomorophic to $\mathbb{Z}^n$, *i.e.*, there exists a reversible mapping function ϕ between $\mathbb{Z}^n$ and M. For a homomorphism ϕ from a $\mathbb{Z}$-module M to a $\mathbb{Z}$-module N ($\phi : M \to N$), the kernel of ϕ, denoted as $Ker(\phi)$, is defined as

$$Ker(\phi) := \{m \in M | \phi(m) = 0 \in N\}$$

The structure of a finitely generated $\mathbb{Z}$-module M can be represented by two finite sets S and S_K, where S is a set of generators of M for homomorphism ϕ_M, and S_K represents the generator of the kernel of ϕ_M.

$$\mathbb{Z}^M[S] \overset{\phi_M}{\to} M \to 0; \mathbb{Z}^M[S_K] = Ker(\phi_M);$$

Then the structure of M is given as

$$M \simeq \mathbb{Z}^M[S]/Ker(\phi_M) = \mathbb{Z}^M[S]/\mathbb{Z}[S_K].$$

In the QST model, physical actions are represented by a finitely generated $\mathbb{Z}$-module denoted as physical action module M_{PA}. The structure of M_{PA} can be represented by the following generating set S_{TPA}

$$\Big\{1, i, I^\mu\sigma^\nu\tau^a, \varepsilon i\tau^3, \varepsilon\tau^a, \varepsilon I^{[c_0-a]}\tau^a, \pm\varepsilon, \varepsilon i, -3\varepsilon i, -\varepsilon_I I^c\Big\},$$

and the generating set of the kernel S_{KPA}.

$$S_{KPA} = \Big\{2\varepsilon i\tau^3, 2\varepsilon\tau^a, 2\varepsilon I^{[c_0-a]}\tau^a, \pm 2\varepsilon, \varepsilon i, -6\varepsilon i, -3\varepsilon_I I^c\Big\}.$$

From this, the structure of M_{PA} is determined as

$$M_{PA} \simeq \mathbb{Z}^M[S_{TPA}]/\mathbb{Z}^M[S_{KPA}] \simeq \mathbb{Z}^{50} \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_3^3$$

For physics, the free part ($\mathbb{Z}^{50}$) is important. In the QST model, the 50 bases of this submodule is denoted as elementaly action $d\hat{S}_p^{\mathrm{EA}}(p = 1, \cdots, 50)$. Any macroscopic physical action S is written as

$$S = \sum_{p=1}^{50} n_p d\hat{S}_p^{\mathrm{EA}} (n_p \in \mathbb{Z})$$

It is shown that $n_p \geq 0$, which indicates causality. A paper explaining the QST model is in preparation.

References
1) Y. Akiba, RIKEN Accel. Prog. Rep. **54**, 69 (2021).
2) Y. Akiba, RIKEN Accel. Prog. Rep. **54**, 70 (2021).
3) Y. Akiba, RIKEN Accel. Prog. Rep. **54**, 71 (2021).