DCP2 - Formulation

Antisymmetric Distorted Wave Impulse Approximation Calculations for Composite Particle Scattering

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Nuclear Physics, **A474** (1987) 131-154

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1 Antisymmetric DWIA Transition Amplitudes

1.1 Distorted wave transition amplitude

The distorted wave transition amplitude for inelastic and charge exchange scattering, $A(a,b)B$, can be written as

$$T = < \chi_b^{(-)} Bb \, |V| \, Aa\chi_a^{(+)} >$$

where V is the interaction potential.

1.2 Interaction potential in the impulse approximation

In the impulse approximation[1], the interaction potential becomes the sum of effective nucleon-nucleon potentials involved. The interaction potential can then be written

$$\begin{aligned}
V &= \int dx_1 dx_2 dx_1' dx_2' \; \hat{\rho}_T(x_1', x_1) \; \hat{\rho}_P(x_2', x_2) \; v_{12}(x_1' x_2', x_1 x_2) \\
x_i &\equiv (\vec{r}_i, \sigma_i, \tau_i) \qquad (i = 1, 2) \\
x_i' &: \; x_i \text{ after the exchange of nucleons 1 and 2 has taken place} \\
\hat{\rho}_T(x_1, x_1') &= \hat{\psi}_T^\dagger(x_1)\hat{\psi}_T(x_1') \\
& \quad \hat{\psi}_T^\dagger(x_1), \hat{\psi}_T(x_1'); \text{Nucleon field creation and annihilation operators} \\
& \quad \text{Nonlocal density operator for the target(T) system} \\
\hat{\rho}_P(x_2, x_2') &= \hat{\psi}_P^\dagger(x_2)\hat{\psi}_P(x_2') \\
& \quad \text{Nonlocal density operator for the projectile(P) system} \\
v_{12}(x_1' x_2', x_1 x_2) &= < x_1' x_2' | V^i | x_1 x_2 >
\end{aligned}$$

where the effective interactions V^i, like Love and Franey interaction[2], are responsible for the direct ($i = D$) and exchange ($i = E$) reactions, and are assumed to be local.

$$v_{12}(x_1' x_2', x_1 x_2) = < x_1' x_2' | V^D | x_1 x_2 > + (-)^\ell < x_1' x_2' | V^E | x_1 x_2 > \mathcal{P}_r$$

where $\mathcal{P}^r$ is the exchange operator for the spatial coordinates.

We further consider the exchange term in the knock-on reactions, in which the projectile a interacts with the nucleon in the target and ejects it while being captured itself at the same place where the target nucleon was, say the knock-on exchange term. In this consideration, the interaction potential becomes

$$\begin{aligned}
v_{12} &= v_{12}^D \delta(x_1' - x_1)\delta(x_2' - x_2) + v_{12}^E (-)^\ell \mathcal{P}^r \\
v_{12}^D &\equiv < x_1 x_2 | V^D | x_1 x_2 >, \qquad v_{12}^E \equiv < x_1' x_2' | V^E | x_1 x_2 >
\end{aligned}$$

We also assume that the effective interactions contain both central and tensor terms. The tensor operator[3] can be written as

$$\hat{S}_{12} = 3(\hat{\sigma}_1 \cdot \hat{r})(\hat{\sigma}_2 \cdot \hat{r}) - (\hat{\sigma}_1 \cdot \hat{\sigma}_2) = \sum_q \sqrt{4\pi}\sqrt{\frac{2}{5}} Y_{2q}^* \hat{T}_{2q}$$

where $\hat{T}_{2q}$ is a second rank tensor operator and its matrix elements become

$$\begin{aligned}
< s' m_s' | \hat{T}_{2q} | s m_s > &= (s m_s 2q | s' m_s') \hat{s}^{-1} < s || \hat{T}_2 || s' > \delta(ss')\delta(s1) \quad \text{with } \hat{s} = \sqrt{2s+1} \\
&= (s m_s 2q | s m_s') \sqrt{20} \qquad \text{for } \; s = 1 \\
< s' m_s' | \hat{S}_{12} | s m_s > &= \sum_q \sqrt{4\pi}\sqrt{8} (s m_s 2q | s m_s') Y_{2q}^*
\end{aligned}$$

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[1] G. R. Satchler, "Direct Nuclear Reactions" (1983), Sec. 3.10.4 and 15.2.3

[2] W. G. Love and M. A. Franey, Phys. Rev. C24, 1073 (1981).

[3] D. M. Brink and G. R. Satchler, "Angular Momentum, 2nd Ed. (1968)", Sec. 6.3.3.

1.3 Spacial coordinates chosen

We choose the space coordinates in the direct and exchange processes as follows, which are shown in Fig. 1,

$\vec{r}_a$: Relative coordinate in the initial channel, a and A.
$\vec{r}_b$: Relative coordinate in the final channel after the exchange process, b and B.
$\vec{r}_1$: Position vector measured from the center of mass of the target system consisted of an interacting nucleon and (A-1) nucleus.
$\vec{r}_2$: Position vector measured from the center of mass of the projectile system consisted of an interacting nucleon and (a-1) nucleus.
$\vec{r}_1'$: Position vector measured from the center of mass of the target system consisted of an interacting nucleon in the projectile and (A-1) nucleus after the exchange process.
$\vec{r}_2'$: Position vector measured from the center of mass of the projectile system consisted of an interacting nucleon in the target and (a-1) nucleus after the exchange process.
$\vec{r}$: Distance vector between the two interacting nucleons 1 and 2.

In other words, the primed coordinates are those after the exchange of nucleons 1 and 2 has taken place.

The channel coordinate $\vec{r}_b$ is related to $\vec{r}_a$, as seen in Fig.2 where the coordinates are chosen in the lab system,

$$\begin{aligned}\vec{r}_b &= -\frac{\vec{r}_{1L}'}{m_A}+\frac{\vec{r}_{1L}}{m_A}+\vec{r}_a-\frac{\vec{r}_{2L}}{m_a}+\frac{\vec{r}_{2L}'}{m_a}\\ &= \vec{r}_a+(\frac{1}{m_A}+\frac{1}{m_a})\vec{r}\\ &= \vec{r}_a+\frac{\vec{r}}{\mu}\end{aligned}$$

We set $m_A \to \infty$, and then $\mu \to m_a$. We thus have Eq.(3) in the original USO paper. (Hereafter use it as "[Eq.(3)]".)

$$\vec{r}_a = \vec{r}_b - \frac{\vec{r}}{m_a}$$

Physically, the difference between $\vec{r}_a$ and $\vec{r}_b$ arises because of the interchange of the positions of nucleons 1 and 2 in the exchange process.

In the direct process, the coordinates $(\vec{r}_a, \vec{r}, \vec{r}_1, \vec{r}_2)$ are not independent each other, we choose $(\vec{r}_a, \vec{r}, \vec{r}_1)$ as independent coordinates, and then, as shown in Fig.1,

$$\vec{r}_2 = \vec{r}_2' + \vec{r} = \vec{r}_1 + \vec{r}_a + \vec{r}$$

We will break up $\vec{r}_2$ into $\vec{r}_2'$ and $\vec{r}$ at the first stage and then $\vec{r}_2'$ into $\vec{r}_1$ and $\vec{r}_a$ at the second stage.

In the exchange process, we choose $(\vec{r}_a, \vec{r}_b, \vec{r}_1)$ as independent coordinates, and

$$\begin{aligned}\vec{r}_1' &= \vec{r}_1 + \vec{r}\\ \vec{r}_2' &= \vec{r}_1 + \vec{r}_b\\ \vec{r}_2 &= \vec{r}_2' + \vec{r} = \vec{r}_1 + \vec{r}_a + \vec{r}\\ \vec{r}_b &= \vec{r}_a + \frac{\vec{r}}{m_a}\end{aligned}$$

as seen in Fig.1. At the final stage we break up $\vec{r}$ into $\vec{r}_a$ and $\vec{r}_b$ and need the Jacobian associated with the transformation of the integral variable from $\vec{r}$ to $\vec{r}_b$.

1.4 Antisymmetric DWIA transition amplitudes

The antisymmetric DWIA transition amplitude T, [Eq.(1)] and [Eq.(2)], can be written as

$$\begin{aligned}
T &= T^D + T^E \\
T^D &= \int d\vec{r}_a \int dx_1 \int dx_2 \; \chi_b^{(-)*}(\vec{k}_b, \vec{r}_a) < Bb|v_{12}^D(\vec{r})\hat{\rho}_T(x_1, x_1)\hat{\rho}_P(x_2, x_2)|Aa > \chi_a^{(+)}(\vec{k}_a, \vec{r}_a) \\
T^E &= (-)^\ell \mathcal{P}^r \\
& \quad \times \int d\vec{r}_a \int dx_1 \int dx_2 \; \chi_b^{(-)*}(\vec{k}_b, \vec{r}_b) < Bb|v_{12}^E(\vec{r})\hat{\rho}_T(x_1, x_1')\hat{\rho}_P(x_2, x_2')|Aa > \chi_a^{(+)}(\vec{k}_a, \vec{r}_a)
\end{aligned}$$

1.5 Target nuclear density matrix element

In the j-representation, the nucleon field creation and annihilation operators, [Eq.(6a)], can be written, respectively,

$$\begin{aligned}
\hat{\psi}_T^\dagger(x_1) &= \sum_{p,\nu_p} \hat{a}^\dagger_{j_p m_p \nu_p} \phi_{j_p m_p} \eta_{\nu_p} \\
\hat{\psi}_T(x_1) &= \sum_{h,\nu_h} \hat{a}_{j_h m_h \nu_h} \phi^*_{j_h m_h} \eta^*_{\nu_h}
\end{aligned}$$

$\hat{a}^\dagger_{j_p m_p \nu_p}(\hat{a}_{j_h m_h \nu_h})$: single-particle (hole) creation (annihilation) operator

$\phi_{j_p m_p}(\phi_{j_h m_h})$: single-particle (hole) wave function

$\eta_{\nu_p}(\eta_{\nu_h})$: isospin part of the single particle (hole) wave function

The target density operator, [Eq.(5a)], can be written

$$\begin{aligned}
\hat{\rho}_T(x_1, x_1') &\equiv \hat{\psi}_T^\dagger(x_1)\hat{\psi}_T(x_1') \\
&= \sum_{ph,\nu_p\nu_h} \hat{a}^\dagger_{j_p m_p \nu_p} \hat{a}_{j_h m_h \nu_h} \phi_{j_p m_p} \phi^*_{j_h m_h} \eta_{\nu_p} \eta^*_{\nu_h} \\
&= \sum_{ph,\nu_p\nu_h,j_t m_t t_1 \nu_1} (-)^{(1/2-\nu_h+j_h-m_h)} (\frac{1}{2}\nu_p \frac{1}{2}, -\nu_h|t_1\nu_1)(j_p m_p j_h, -m_h|j_t m_t) \\
& \qquad \hat{a}^\dagger_{j_p m_p \nu_p} \hat{a}_{j_h m_h \nu_h} [\phi_{j_p} \phi^*_{\tilde{j}_h}]_{j_t m_t} [\eta_1 \eta^*_{\tilde{2}}]_{t_1 \nu_1}
\end{aligned}$$

where we use

$$\begin{aligned}
\phi_{j_p m_p} \phi^*_{j_h m_h} &= \sum_{j_t m_t} (-)^{j_h - m_h} (j_p m_p j_h, -m_h|j_t m_t) [\phi_{j_p} \phi^*_{\tilde{j}_h}]_{j_t m_t} \\
\eta_{\nu_p} \eta^*_{\nu_h} &= \sum_{t_1 \nu_1} (-)^{1/2-\nu_h} (\frac{1}{2}\nu_p \frac{1}{2}, -\nu_h|t_1\nu_1)[\eta_1 \eta^*_{\tilde{2}}]_{t_1 \nu_1}
\end{aligned}$$

and the tilde on the top of hole states indicates the time reversed state with the time reversal convention,

$$\phi_{\tilde{jm}} \equiv (-)^{j+m} \phi_{j,-m} = (-)^j \phi^*_{jm}, \qquad \phi^*_{\tilde{jm}} = (-)^j \phi_{jm}$$

We further define

$$\begin{aligned}
\mathcal{A}^\dagger_{j_p j_{\tilde{h}} j_t m_t \nu_p \tilde{\nu}_h} &\equiv \sum_{m_p m_h} (-)^{j_h - m_h} (j_p m_p j_h, -m_h|j_t m_t) \hat{a}^\dagger_{j_p m_p \nu_p} \hat{a}_{j_h m_h \nu_h} \\
&= [\hat{a}^\dagger_{j_p \nu_p} \hat{a}_{j_{\tilde{h}} \tilde{\nu}_h}]_{j_t m_t}
\end{aligned}$$

The target density operator then becomes

$$\hat\rho_T(x_1,x_1') = \sum_{j_pj_hj_tm_t\nu_p\nu_ht_1\nu_1} \beta_{\nu_p\nu_h}\mathcal{A}^\dagger_{j_pj_{\tilde h}j_tm_t\nu_p\tilde\nu_h}[\phi_{j_p}\phi^*_{\tilde j_h}]_{j_tm_t}[\eta_1\eta^*_{\tilde 2}]_{t_1\nu_1}$$

$$\beta_{\nu_p\nu_h} \equiv (-)^{1/2-\nu_h}(\frac{1}{2}\nu_p\frac{1}{2},-\nu_h|t_1\nu_1)$$

The values of $\beta_{\nu_p\nu_h}$ are following,

t_1	ν_1	ν_p	ν_h	β	
1	1	1/2	-1/2	-1	Charge exchange (pn^{-1})
1	0	1/2	1/2	$1/\sqrt{2}$	Inelastic (pp^{-1})
1	0	-1/2	-1/2	$-1/\sqrt{2}$	Inelastic (nn^{-1})
1	-1	-1/2	1/2	1	Charge exchange (np^{-1})
0	0	1/2	1/2	$1/\sqrt{2}$	Inelastic(pp^{-1})
0	0	-1/2	-1/2	$1/\sqrt{2}$	Inelastic (nn^{-1})

Now the matrix element of $\mathcal{A}$ in the target system is written as

$$< B|\mathcal{A}^\dagger_{j_pj_{\tilde h}j_tm_t\nu_p\tilde\nu_h}|A> = \hat I_B^{-1}(I_AM_Aj_tm_t|I_BM_B) < I_B||[\hat a^\dagger_{j_p\nu_p}\hat a_{\tilde{j_h\nu_h}}]_{j_t}||I_A>$$

where $\hat I = \sqrt{2I+1}$ and $< I_B||[\hat a^\dagger_{j_p\nu_p}\hat a_{\tilde{j_h\nu_h}}]_{j_t}||I_A>$ is the spectroscopic amplitude.

We now change jj coupling scheme to ℓs coupling[4] in $[\phi_{j_p}\phi^*_{\tilde j_h}]_{j_tm_t}$,

$$[\phi_{j_p}\phi^*_{\tilde j_h}]_{j_tm_t} = \sum_{\ell_1m_{\ell_1}s_1m_1}(\ell_1m_{\ell_1}s_1m_1|j_tm_t)X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1s_1j_t)[\phi_{\ell_p}\phi^*_{\tilde\ell_h}]_{\ell_1m_{\ell_1}}[\xi_1\xi^*_{\tilde 2}]_{s_1m_1}$$

$$X = \hat j_p\hat j_h\hat\ell_1\hat s_1\begin{Bmatrix}\ell_p & \frac{1}{2} & j_p\\ \ell_h & \frac{1}{2} & j_h\\ \ell_1 & s_1 & j_t\end{Bmatrix}$$

$$[\xi_1\xi^*_{\tilde 2}]_{s_1m_1} = \sum_{\mu_p\mu_h}(-)^{1/2-\mu_h}(\frac{1}{2}\mu_p\frac{1}{2}\mu_h|s_1m_1)\xi_{1\mu_p}\xi^*_{\tilde 2\mu_h} \qquad ([\text{Eq.(8b)}])$$

ξ : spin part of the single particle wave function

We put them together to obtain the target density matrix element, [Eq.(7)] and [Eq.(8a)],

$$< B|\hat\rho_T(x_1,x_1')|A> = \sum_{j_t\ell_1s_1t_1m_{\ell_1}m_1}\hat I_B^{-1}(I_AM_Aj_tm_t|I_BM_B)(\ell_1m_{\ell_1}s_1m_1|j_tm_t)\rho^{t_1\nu_1}_{T,\ell_1m_{\ell_1}}(\vec r_1,\vec r_1{}')[\xi_1\xi^*_{\tilde 2}]_{s_1m_1}[\eta_1\eta^*_{\tilde 2}]_{t_1\nu_1}$$

$$\rho^{t_1\nu_1}_{T,\ell_1m_{\ell_1}}(\vec r_1,\vec r_1{}') = \sum_{ph,\nu_p\nu_h}\beta_{\nu_p\nu_h}X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1s_1j_t) < I_B||[\hat a^\dagger_{j_p\nu_p}\hat a_{\tilde{j_h\nu_h}}]_{j_t}||I_A>[\phi_{\ell_p}(\vec r_1)\phi^*_{\tilde\ell_h}(\vec r_1{}')]_{\ell_1m_{\ell_1}}$$

[4]A. deShalit and H. Feshbach, ”Theoretical Nuclear Physics, Vol.I (1974)”, Sec. V.3

The expression for the diagonal matrix element $\hat{\rho}_T(x_1, x_1)$ can be obtained by replacing by $\vec{r}_1', \xi(2), \eta(2)$ through $\vec{r}_1, \xi(1), \eta(1)$, i.e.,

$$\begin{aligned}
\rho^{t_1\nu_1,D}_{T,\ell_1 m_{\ell_1}}(\vec{r}_1) &= \sum_{ph,\nu_p\nu_h} \beta_{\nu_p\nu_h} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \\
&\qquad < I_B||[\hat{a}^\dagger_{j_p\nu_p} \hat{a}_{\tilde{j_h}\tilde{\nu_h}}]_{j_t}||I_A > [\phi_{\ell_p}(\vec{r}_1)\phi^*_{\tilde{\ell}_h}(\vec{r}_1)]_{\ell_1 m_{\ell_1}} \\
&= \sum_{ph,\nu_p\nu_h} \beta_{\nu_p\nu_h} \rho^D_{T,\ell_1 m_{\ell_1}} \\
&\equiv \sum_{ph,\nu_p\nu_h} \beta_{\nu_p\nu_h} \frac{1}{\sqrt{4\pi}} \rho^D_{T,\ell_1} Y_{\ell_1 m_1}(\hat{r}_1) \\
\rho^D_{T,\ell_1}(r_1) &= X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B||[\hat{a}^\dagger_{j_p\nu_p} \hat{a}_{\tilde{j_h}\tilde{\nu_h}}]_{j_t}||I_A > \sqrt{4\pi} d_{\ell_p \ell_h \ell_1} R_{\ell_p}(r_1) R_{\ell_h}(r_1)
\end{aligned}$$

The term ρ^D_{T,ℓ_1} is calculated in SUBROUTINE FFCALD, saved as TRHO(r_1, ℓ_1).

Here we have angular momentum coupling relations,

$$\vec{\ell}_p + \vec{\ell}_h = \vec{\ell}_1, \qquad \vec{j}_p + \vec{j}_h = \vec{j}_t, \qquad \vec{\ell}_1 + \vec{s}_1 = \vec{j}_t$$
$$s_1 = 0 \quad \text{or} \quad 1$$

1.6 Projectile nuclear density matrix element

We assume that the space part of the projectile wave function is in the s-state as is the case for scattering induced by deuteron, ^3He, triton and alpha particles. In the m-representation, the nucleon field creation and annihilation operators, [Eq.(6b)], can be written, respectively,

$$\begin{aligned}
\hat{\psi}^\dagger_P(x_2) &= \sum_{im\nu} \hat{c}^\dagger_i \hat{c}^\dagger_\mu \hat{c}^\dagger_\nu \phi_i(\vec{r}_2) \xi_\mu(2) \eta_\nu(2) \\
\hat{\psi}_P(x_2) &= \sum_{i\mu\nu} \hat{c}_i \hat{c}_\mu \hat{c}_\nu \phi^*_i(\vec{r}_2) \xi^*_\mu(2) \eta^*_\nu(2)
\end{aligned}$$

ϕ_i : spatial part of projectile wave function with $(i = \ell_2 m_2)$
ξ_μ : spin part
η_ν : isospin part
$\hat{c}^\dagger(\hat{c})$: creation (annihilation) operator with (i, μ, ν)

The projectile density operator can be written

$$\begin{aligned}
\hat{\rho}_P(x_2, x_2') &\equiv \hat{\psi}^\dagger_P(x_2) \hat{\psi}_P(x_2') \\
&= \sum \hat{c}^\dagger_i \hat{c}_i \phi_i(\vec{r}_2) \phi^*_i(\vec{r}_2') \ \hat{c}^\dagger_\mu \hat{c}_\mu \xi_\mu(2) \xi^*_\mu(2') \ \hat{c}^\dagger_\nu \hat{c}_\nu \eta_\nu(2) \eta^*_\nu(2')
\end{aligned}$$

We assume that the spacial part of the wave function is scalar, and thus $\ell = 0$, and that after the exchange, the spin and isospin parts of $2'$ become those of the particle 1 (knock-on exchange).

$$\begin{aligned}
\hat{\rho}_P(x_2, x_2') &= \sum_{\ell_2} [\hat{c}^\dagger_{\ell_2} \hat{c}_{\tilde{\ell}_2}]_{00} [\phi_{\ell_2}(\vec{r}_2) \phi_{\tilde{\ell}_2}(\vec{r}_2')]_{00} \\
&\qquad \sum_{s_2 m_2} [\hat{c}^\dagger \tilde{\hat{c}}]_{s_2 m_2} [\xi_2 \xi^*_1]_{s_2 m_2} \sum_{t_2\nu_2} [\hat{c}^\dagger \tilde{\hat{c}}]_{t_2\nu_2} [\eta_2 \eta^*_1]_{t_2\nu_2}
\end{aligned}$$

where we use

$$\phi_{\ell_2}(\vec{r}_2) \phi_{\tilde{\ell}_2}(\vec{r}_2') = (-)^{\ell_2} (\ell_2 m_2 \ell_2, -m_2|00) [\phi_{\ell_2}(\vec{r}_2) \phi_{\tilde{\ell}_2}(\vec{r}_2')]_{00}$$

$$\sum_{m_2}(-)^{\ell_2}(\ell_2 m_2 \ell_2,-m_2|00)\hat{c}^\dagger_{\ell_2}\hat{c}_{\ell_2} = [\hat{c}^\dagger_{\ell_2}\hat{c}_{\tilde{\ell}_2}]_{00}$$

$$\xi_{\mu_2}\xi^*_{\mu_1} = \sum_{sm}(-)^{1/2-\mu_1}(\frac{1}{2}\mu_2\frac{1}{2},-\mu_1|sm)[\xi_2\xi^*_{\tilde{1}}]_{sm}$$

$$\sum_{\mu_1\mu_2}(-)^{1/2-\mu_1}(\frac{1}{2}\mu_2\frac{1}{2},-\mu_1|sm)\hat{c}^\dagger_{\mu_2}\hat{c}_{\mu_1} = [\hat{c}^\dagger\tilde{\hat{c}}]_{sm}$$

$$\eta_{\nu_2}\eta^*_{\nu_1} = \sum_{t\nu}(-)^{1/2-\nu_1}(\frac{1}{2}\nu_2\frac{1}{2},-\nu_1|t\nu)[\eta_2\eta^*_{\tilde{1}}]_{t\nu}$$

$$\sum_{\nu_1\nu_2}(-)^{1/2-\nu_1}(\frac{1}{2}\nu_2\frac{1}{2},-\nu_1|t\nu)\hat{c}^\dagger_{\nu_2}\hat{c}_{\nu_1} = [\hat{c}^\dagger\tilde{\hat{c}}]_{t\nu}$$

The spatial part of the projectile density matrix element, [Eq.(11)], is

$$\rho_P(\vec{r}_2,\vec{r}'_2) \equiv \sum_{\ell_2} < b||[\hat{c}^\dagger_{\ell_2}\hat{c}_{\tilde{\ell}_2}]_{00}||a > [\phi_{\ell_2}(\vec{r}_2)\phi_{\tilde{\ell}_2}(\vec{r}'_2)]_{00}$$

The spin-isospin part of the matrix element becomes

$$< b|[...]|a > = \hat{s}_b^{-1}(s_a m_a s_2 m_2|s_b m_b) < s_b t_b|||[...]|||s_a t_a >$$

We put them together to obtain the projectile nuclear density matrix element, [Eq.(10)]

$$\begin{aligned} < b|\hat{\rho}_P(x_2,x'_2)|a > &= \sum_{s_2 m_2 t_2 \nu_2} \hat{s}_b^{-1}(s_a m_a s_2 m_2|s_b m_b) < b||[c^\dagger c]_{s_2 t_2 \nu_2}]||a > \\ & [\xi_2\xi_{\tilde{1}}]_{s_2 m_2}[\eta_2\eta_{\tilde{1}}]_{t_2\nu_2}\rho_P(\vec{r}_2,\vec{r}'_2) \end{aligned}$$

The expression for the diagonal matrix element $\hat{\rho}_P(x_2,x_2)$ can be obtained by replacing by $\vec{r}'_2,\xi(1),\eta(1)$ through $\vec{r}_2,\xi(2),\eta(2)$.

Here we have angular momentum coupling relations,

$$\vec{s}_a+\vec{s}_b=\vec{s}_2, \qquad s_2=0 \quad \text{or} \quad 1$$

1.7 Spin and isospin parts of the NN interaction

1) Spin part of transition amplitudes

The spin part of transition amplitude can be written

$$\begin{aligned} v^{spin} &= < [\xi_1\xi^*_{\tilde{2}}]_{s_1 m_1}|v_{12}|[\xi_2\xi_{\tilde{1}}]_{s_2 m_2} > \\ &= \sum_{\sigma}(-)^{1/2+\sigma_2+1/2+\sigma_1}(\frac{1}{2}\sigma_1\frac{1}{2},-\sigma'_2|s_1 m_1)(\frac{1}{2}\sigma_2\frac{1}{2},-\sigma'_1|s_2 m_2) \\ & \sum_{s m_s m_s}(\frac{1}{2}\sigma_1\frac{1}{2}\sigma_2|s m_s)(\frac{1}{2}\sigma'_2\frac{1}{2}\sigma'_1|s m'_s) < s m_s|v_{12}|s m'_s > \end{aligned}$$

where

$$\begin{aligned} < s m_s|v_{12}|s m'_s > &= \sum_{kq} f_k(s m_s k q|s m'_s)Y^*_{kq}(\hat{r})v_{sk}(r) \\ &= \sqrt{4\pi}\sum_{kq} f_k(-)^{s-m_s}\hat{s}\hat{k}^{-1}(-)^q(s m_s s m'_s|k,-q)Y^*_{kq}(\hat{r})v_{sk}(r) \end{aligned}$$

with $f_0=1$ for the central part and $f_2=\sqrt{8}$ for the tensor part.

Summing 5 CG coefficients over σ and m's yields

$$\begin{aligned}
Sum &= \sum_{\sigma,m}(-)^{1/2+\sigma_2+1/2+\sigma_1}(-)^{s-m_s}(\frac{1}{2}\sigma_1\frac{1}{2},-\sigma_2'|s_1m_1)(\frac{1}{2}\sigma_2\frac{1}{2},-\sigma_1'|s_2m_2) \\
&\qquad (\frac{1}{2}\sigma_1\frac{1}{2}\sigma_2|sm_s)(\frac{1}{2}\sigma_2'\frac{1}{2}\sigma_1'|sm_s')(sm_ssm_s'|k,-q) \\
&= (s_1m_1s_2m_2|k,-q)X(\frac{1}{2}\frac{1}{2}s,\frac{1}{2}\frac{1}{2}s;s_1s_2k) \\
&= (s_1m_1s_2m_2|k,-q)\hat{s}^2\hat{s}_1^2\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ \frac{1}{2} & \frac{1}{2} & s \\ s_1 & s_1 & k \end{Bmatrix}
\end{aligned}$$

Thus we have the spin part of transition amplitude,

$$v^{spin} = \sum_{kq}\sqrt{4\pi}f_k\hat{k}^{-1}(-)^q\delta(s_1s_2)(s_1m_1s_2m_2|k,-q)\hat{s}^3\hat{s}_1^2\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ \frac{1}{2} & \frac{1}{2} & s \\ s_1 & s_1 & k \end{Bmatrix}Y_{kq}^*(\hat{r})v_{sk}(r)$$

Here we have angular momentum coupling relations,

$$\vec{s}_1+\vec{s}_2=\vec{k}, \qquad k=0 \quad \text{or} \quad 2$$

2) Isospin part of transition amplitudes

The isospin part can be obtained by setting $k=0$ and replacing spins by isospins in the spin part and thus leads

$$\begin{aligned}
v^{isospin} &= \delta(t_1t_2)(t_1\nu_1t_1,-\nu_1|00)\hat{t}^3\hat{t}_1^2\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ \frac{1}{2} & \frac{1}{2} & t \\ t_1 & t_1 & 0 \end{Bmatrix}v_t(r) \\
&= \delta(t_1t_2)(-)^{t_1-\nu_1}\hat{t}^3\hat{t}_1\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ \frac{1}{2} & \frac{1}{2} & t \\ t_1 & t_1 & 0 \end{Bmatrix}v_t(r)
\end{aligned}$$

1.8 Direct and exchange transition amplitudes

The direct ($i=D$) and exchange($i=E$) transition amplitudes become

$$\begin{aligned}
T^i &= \sum \hat{I}_B^{-1}(I_AM_Aj_tm_t|I_BM_B)(\ell_1m_{\ell_1}s_1m_1|j_tm_t)\hat{s}_b^{-1}(s_am_as_2m_2|s_bm_b) \\
&\times <b||[c^\dagger c]_{s_2m_2t_2\nu_2}]||a> \sqrt{4\pi}f_k\hat{k}^{-1}(-)^q\delta(s_1s_2)(s_1m_1s_2m_2|k,-q)\hat{s}^3\hat{s}_1^2 \\
&\times \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ \frac{1}{2} & \frac{1}{2} & s \\ s_1 & s_1 & k \end{Bmatrix}\delta(t_1t_2)(-)^{t_1-\nu_1}\hat{t}^3\hat{t}_1\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ \frac{1}{2} & \frac{1}{2} & t \\ t_1 & t_1 & 0 \end{Bmatrix} \\
&\times \int d\vec{r}_a\int d\vec{r}_1\int d\vec{r}\;\chi_b^{(-)*}(\vec{k}_b,\vec{r}_b)v_{stk}^i(r)Y_{kq}^*(\hat{r})\rho_{T,\ell_1m_{\ell_1}}^{t_1\nu_1}(\vec{r}_1,\vec{r}_1')\rho_P(\vec{r}_2,\vec{r}_2')\chi_a^{(+)}(\vec{k}_a,\vec{r}_a)
\end{aligned}$$

where we sum over $j_t\ell_1s_1t_1s_2t_2k$ and their z-components.

For the spatial part of the direct transition amplitudes, we need to change the coordinates such that

$$\rho_P(\vec{r}_2,\vec{r}_2')=\rho_P^D(\vec{r}_2,\vec{r}_2)\rightarrow\rho_P^D(\vec{r}_2',\vec{r})\rightarrow\rho_P^D(\vec{r}_b,\vec{r}_1,\vec{r})$$

while for the spatial exchange part, we change the coordinates such that

$$\begin{array}{rcl} \rho_T^E(\vec{r}_1,\vec{r}_1') & \rightarrow & \rho_T^E(\vec{r}_1,\vec{r}) \\ \rho_P^E(\vec{r}_2,\vec{r}_2') & \rightarrow & \rho_P^E(\vec{r}_2',\vec{r}) \ \rightarrow \ \rho_P^E(\vec{r}_b,\vec{r}_1,\vec{r}) \end{array}$$

which will be done in the form factor calculations. (See Section 2.)

We first couple $\rho^{t_1\nu_1}_{T,\ell_1 m_{\ell_1}}$ and $Y^*_{kq}(\hat{r})$ which gives

$$\rho^{t_1\nu_1}_{T,\ell_1 m_{\ell_1}} Y^*_{kq}(\hat{r}) = \sum_{\ell m_{\ell_t}} (-)^q (\ell_1 m_{\ell_1} k,-q|\ell_t m_{\ell_t})[\rho^{t_1\nu_1}_{T,\ell_1 m_{\ell_1}} Y^*_{kq}(\hat{r})]_{\ell_t m_{\ell_t}}$$

We combine 3 CG's, considering $s_1 = s_2 = s_t$,

$$\begin{array}{rcl} 3\ \mathrm{CG} & = & \displaystyle\sum_{m_t} (\ell_1 m_{\ell_1} s_t m_t|j_t m_t)(s_t m_t s_t m_t|k,-q)(\ell_1 m_{\ell_1} k,-q|\ell_t m_{\ell_t}) \\ & = & \hat{j}_t \hat{k} (s_t m_t j_t m_{j_t}|\ell_t m_{\ell_t}) W(s_t \ell_t s_t \ell_1; j_t k) \end{array}$$

We further use

$$\begin{array}{rcl} \hat{I}_B^{-1}(I_A M_A j_t m_t|I_B M_B) & = & (-)^{I_A - M_A}(I_A M_A I_B,-M_B|j_t,-m_{j_t})\hat{j}_t^{-1} \\ \hat{s}_b^{-1}(s_a m_a s_t m_t|s_b m_b) & = & (-)^{s_a - m_a}(s_a m_a s_b,-m_b|s_t,-m_t)\hat{s}_t^{-1} \end{array}$$

We finally obtain the transition amplitudes $T^i (i = D, E)$ of [Eq.(12)],

$$\begin{array}{rcl} T^i & = & \displaystyle\sum_{j_t s_t \ell_t m_{\ell_t}} (-)^{I_A - M_A}(I_A M_A I_B,-M_B|j_t,-m_{j_t})(-)^{s_a - m_a}(s_a m_a s_b,-m_b|s_t,-m_t) \\ & & \times\ \displaystyle (s_t m_t j_t m_{j_t}|\ell_t m_{\ell_t}) \sum_{k\ell_1 t_1} \alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t} T^i_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} \end{array}$$

where $(s_t m_t j_t m_{j_t}|\ell_t m_{\ell_t})$ is calculated in the MAIN program and stored as TFAC(m_t, m_{j_t}, ℓ_t). The expansion coefficient $\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t}$, [Eq.(13)], is

$$\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t} = W(s_t \ell_t s_t \ell_1; j_t k)\hat{s}_t^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a >$$

The details of α coefficients are presented in Chapter IV for several different reactions. SUBROUTINE AFACAL calculates coefficients and stores as ALPHA(lsk, ℓ_t). We define the force components, [Eq.(16)],

$$V^i_{t_1 s_1 k}(r) = \sqrt{4\pi} f_k \hat{s}_1^2 \hat{t}_1^2 \sum_{st} \hat{s}^3 \hat{t}^3 \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s \\ \frac{1}{2} & \frac{1}{2} & s \\ s_1 & s_1 & k \end{array} \right\} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & t \\ \frac{1}{2} & \frac{1}{2} & t \\ t_1 & t_1 & 0 \end{array} \right\} P_i v^i_{tsk}(r)$$

Where $P_D = 1$ and $P_E = (-)^{s+t+1}$, while $f_0 = 1$ and $f_2 = \sqrt{8}$. The interaction potential $V^i_{t_1 s_1 k}(r)$ is calculated in the SUBROUTINE EFFINT(D,E) and stored as VV$(t_1 s_1 k, r_a)$. The number of elements of $\{tsk\}$ is 6 and real and imaginary parts make 12.

Here we have angular momentum coupling relations,

$$\vec{\ell}_1 + \vec{\ell}_t = \vec{k}, \qquad \vec{\ell}_t + \vec{s}_t = \vec{j}_t,$$
$$s_1 = s_2 = s_t$$

The direct and exchange transition amplitudes, [Eq.(14)], are then given by

$$\begin{array}{rcl} T^D_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} & = & \displaystyle\int d\vec{r}_a \chi_b^{(-)*}(\vec{k}_b,\vec{r}_a) F^D_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a)\chi_a^{(+)}(\vec{k}_a,\vec{r}_a), \\ T^E_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} & = & \displaystyle\int d\vec{r}_a \int d\vec{r}_b \chi_b^{(-)*}(\vec{k}_b,\vec{r}_b) F^E_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b,\vec{r}_a)\chi_a^{(+)}(\vec{k}_a,\vec{r}_a) \end{array}$$

with the direct and exchange form factors, [Eq.(15)],

$$\begin{aligned} F^D_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a) &= \int d\vec{r}_1 \int d\vec{r}_2 \rho^D_P(\vec{r}_2) V^D_{t_1 s_1 k}(r) [\rho^D_{T,\ell_1}(\vec{r}_1) Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \\ F^E_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b, \vec{r}_a) &= J \int d\vec{r}_1 \rho^E_P(\vec{r}_2, \vec{r}_2') V^E_{t_1 s_1 k}(r) [\rho^E_{T,\ell_1}(\vec{r}_1, \vec{r}_1') Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \end{aligned}$$

where J is the Jacobian associated with the transformation of the integral variable from $\vec{r}$ to $\vec{r}_a$ and $\vec{r}_b$. (See Section 1.3.)

1.9 Partial wave expansions

We now expand the distorted waves $\chi^{\pm}$, [Eq.(17)], into partial waves. We choose the coordinate system such that the initial projectile momentum $\vec{k}_a$ points along the z-axis:

$$\begin{aligned} \chi^{(+)}_a(\vec{k}_a, \vec{r}_a) &= \frac{\sqrt{4\pi}}{k_a r_a} \sum_{\ell_a} i^{\ell_a} \hat{\ell}_a \chi_{\ell_a}(r_a) Y_{\ell_a 0}(\hat{r}_a) \\ \chi^{(-)}_b(\vec{k}_b, \vec{r}_b) &= \frac{4\pi}{k_b r_b} \sum_{\ell_b m_{\ell_b}} i^{-\ell_b} \chi_{\ell_b}(r_b) Y_{\ell_b m_{\ell_b}}(\hat{r}_b) Y^*_{\ell_b m_{\ell_b}}(\hat{k}_b) \end{aligned}$$

We can rewrite the transition amplitudes, [Eq.(18)], as (See below for the proof.)

$$T^i_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} = \frac{(4\pi)^{3/2}}{k_a k_b} \sum_{\ell_a \ell_b} i^{\ell_a - \ell_b + \pi} \hat{\ell}_a (\ell_a 0 \ell_b m_{\ell_t} | \ell_t m_{\ell_t}) O^i_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} Y_{\ell_b m_{\ell_t}}(\hat{k}_b)$$

where the overlap integral O^i denotes the radial integral, [Eq.(19)], defined by

$$\begin{aligned} O^D_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} &= d_{\ell_a \ell_b \ell_t} \int dr_a \chi_{\ell_b}(r_a) f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) \chi_{\ell_a}(r_a) \\ O^E_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} &= J \int dr_b \int dr_a r_b r_a \chi_{\ell_b}(r_b) f^E_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b}(r_b, r_a) \chi_{\ell_a}(r_a) \end{aligned}$$

with a help of C.2, and d-factor is defined as

$$d_{\ell_a \ell_b \ell_t} \equiv \frac{1}{\sqrt{4\pi}} \hat{\ell}_a \hat{\ell}_b \hat{\ell}_t^{-1} (\ell_a 0 \ell_b 0 | \ell_t 0)$$

Here we have angular momentum coupling relations,

$$\vec{\ell}_a + \vec{\ell}_b = \vec{\ell}_t, \qquad \ell_a + \ell_b + \ell_t = \text{ even}$$

The factors $f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a)$ and $f^E_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b}(r_b, r_a)$ are the radial direct and exchange form factors, [Eq.(20)], defined by, respectively,

$$\begin{aligned} f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi} \int d\hat{r}_a Y^*_{\ell_t m_{\ell_t}}(\hat{r}_a) F^D_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a) \\ f^E_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b}(r_b, r_a) &= i^{-\pi} \int d\hat{r}_b \int d\hat{r}_a [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^*_{\ell_t m_{\ell_t}} F^E_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b, \vec{r}_a) \end{aligned}$$

Note that the phase factor $i^{-\pi}$ is rather arbitrary, but makes radial form factor a real quantity. The phase is chosen such that $\pi = 0$ or 1 depending on whether the parity is changed in the reaction or not.

The proof of the transition amplitudes expressed in terms of radial overlap integrals can obtained in a straightforward way.

1) Direct part

$$\begin{aligned}
T^{D}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} &= \int d\vec{r}_a \chi_b^{(-)*}(\vec{k}_b,\vec{r}_a) F^{D}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a)\chi_a^{(+)}(\vec{k}_a,\vec{r}_a), \\
&= \frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\int dr_a \chi_{\ell_b}(r_a) f^{D}_{t_1 s_1 \ell_1 k \ell_t}(r_a)\chi_{\ell_a}(r_a) \\
&\qquad \times \int d\hat{r}_a Y^{*}_{\ell_b m_{\ell_b}}(\hat{r}_a) Y_{\ell_t m_{\ell_t}}(\hat{r}_a) Y_{\ell_a 0}(\hat{r}_a) \;\times\; \hat{\ell}_a Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
&= \frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\int dr_a \chi_{\ell_b}(r_a) f^{D}_{t_1 s_1 \ell_1 k \ell_t}(r_a)\chi_{\ell_a}(r_a) \\
&\qquad \times\; d_{\ell_a\ell_t\ell_b}(\ell_a 0 \ell_t m_{\ell_t}|\ell_b m_{\ell_t})\; \hat{\ell}_a Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
&= \frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\hat{\ell}_a(\ell_a 0 \ell_b m_{\ell_t}|\ell_t m_{\ell_t}) O^{i}_{t_1 s_1 \ell_1 k \ell_t,\ell_a\ell_b} Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
O^{D}_{t_1 s_1 \ell_1 k \ell_t,\ell_a\ell_b} &= d_{\ell_a\ell_b\ell_t}\int dr_a \chi_{\ell_b}(r_a) f^{D}_{t_1 s_1 \ell_1 k \ell_t}(r_a)\chi_{\ell_a}(r_a)
\end{aligned}$$

since

$$\begin{aligned}
d_{\ell_a\ell_t\ell_b}(\ell_a 0 \ell_t m_{\ell_t}|\ell_b m_{\ell_t}) &= \frac{1}{\sqrt{4\pi}}\hat{\ell}_a\hat{\ell}_t\hat{\ell}_b^{-1}(-)^{\ell_a}\hat{\ell}_b\hat{\ell}_t^{-1}(\ell_a 0 \ell_b 0|\ell_t 0)(-)^{\ell_a}\hat{\ell}_b\hat{\ell}_t^{-1}(\ell_a 0 \ell_b m_{\ell_t}|\ell_t m_{\ell_t}) \\
&= d_{\ell_a\ell_b\ell_t}(\ell_a 0 \ell_b m_{\ell_t}|\ell_t m_{\ell_t})
\end{aligned}$$

2) Exchange part

$$\begin{aligned}
T^{E}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} &= \int d\vec{r}_a \int d\vec{r}_b \chi_b^{(-)*}(\vec{k}_b,\vec{r}_b) F^{E}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b,\vec{r}_a)\chi_a^{(+)}(\vec{k}_a,\vec{r}_a) \\
&= J\frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\int dr_a dr_b r_a r_b \chi_{\ell_b}(r_b) f^{E}_{t_1 s_1 \ell_1 k \ell_t}(r_a,r_b)\chi_{\ell_a}(r_a) \\
&\qquad \times \int d\hat{r}_a d\hat{r}_b Y^{*}_{\ell_b m_{\ell_b}}(\hat{r}_b)[Y_{L_a}(\hat{r}_a)Y_{L_b}(\hat{r}_b)]^{*}_{\ell_t m_{\ell_t}} Y_{\ell_a 0}(\hat{r}_a) \;\times\; \hat{\ell}_a Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
&= J\frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\int dr_a dr_b r_a r_b \chi_{\ell_b}(r_b) f^{E}_{t_1 s_1 \ell_1 k \ell_t}(r_a,r_b)\chi_{\ell_a}(r_a) \\
&\qquad \times (L_a M_a L_b M_b|\ell_t m_{\ell_t})\delta(\ell_a L_a)\delta(M_a 0)\delta(\ell_b L_b)\delta(m_{\ell_t},M_b)\; \hat{\ell}_a Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
&= \frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a\ell_b} i^{\ell_a-\ell_b+\pi}\hat{\ell}_a(\ell_a 0 \ell_b m_{\ell_t}|\ell_t m_{\ell_t}) O^{i}_{t_1 s_1 \ell_1 k \ell_t,\ell_a\ell_b} Y_{\ell_b m_{\ell_t}}(\hat{k}_b) \\
O^{E}_{t_1 s_1 \ell_1 k \ell_t,\ell_a\ell_b} &= J\int dr_b \int dr_a r_b r_a \chi_{\ell_b}(r_b) f^{E}_{t_1 s_1 \ell_1 k \ell_t,\ell_a\ell_b}(r_b,r_a)\chi_{\ell_a}(r_a)
\end{aligned}$$

Thus the transition amplitudes of [Eq.(18)] and [Eq.(19)] should be modified in this way, namely, $(\ell_a 0 \ell_t m_{\ell_t}|\ell_b m_{\ell_t})$ to $(\ell_a 0 \ell_b m_{\ell_t}|\ell_t m_{\ell_t})$ in [Eq.(18)] and $d_{\ell_a\ell_t\ell_b}$ to $d_{\ell_a\ell_b\ell_t}$ in [Eq.(19a)]. In fact, it does not make any change in the direct part as you see above, but does make a difference in the exchange part obviously.

1.10 Differential cross section

The differential cross section, [Eq.(21)], is given by

$$\frac{d\sigma}{d\Omega} = \frac{\mu_a\mu_b}{(2\pi\hbar^2)^2}\frac{k_b}{k_a}\frac{1}{(2I_A+1)(2s_a+1)}|\sum_i\sum_{k\ell_1 t_1}\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_1 k \ell_t} T^i_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}|^2$$

where $\mu_a(\mu_b)$ is the reduced mass in the incident (exit) channel and i is $i = D$ for direct transitions and $i = E$ for exchange transitions.

We finally summarize the above differential cross sections combining radial form factors in Section 2. The $\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t}$ coefficients and the transition amplitudes are

$$\begin{aligned}
\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t} &= W(s_t\ell_t s_t \ell_1; j_t k)\hat{s}_t^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a > \\
T^i_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} &= \frac{(4\pi)^{3/2}}{k_a k_b}\sum_{\ell_a \ell_b} i^{\ell_a-\ell_b+\pi}\hat{\ell}_a(\ell_a 0 \ell_t m_{\ell_t}|\ell_b m_{\ell_t}) O^i_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} Y_{\ell_b m_{\ell_t}}(\hat{k}_b)
\end{aligned}$$

where the direct overlap integrals are

$$\begin{aligned}
O^D_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} &= d_{\ell_a \ell_t \ell_b}\int dr_a \chi_{\ell_b}(r_a) f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a)\chi_{\ell_a}(r_a) \\
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi}(-)^{\ell_1}\hat{\ell}_t^{-1}\int r^2 dr V^D_{t_1 s_1 k}(r)\int r_1^2 dr_1 \rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r)\rho^D_{T,\ell_1}(r_1) \\
\rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r) &= \frac{2\pi}{\hat{k}^2}\sum_m \hat{\ell}_t(\ell_t 0 \ell_1 m|km)\int \rho^D_{P,k}(r_a,r_1,\mu,r) Y_{km}(\theta_2',0) Y^*_{\ell_1 m}(\theta,0) d\mu \\
\rho^D_{T,\ell_1}(r_1) &= X(\ell_p \tfrac{1}{2} j_p, \ell_h \tfrac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B||[\hat{a}^\dagger_{j_p \nu_p}\hat{a}_{j_h \tilde{\nu}_h}]_{j_t}||I_A > \sqrt{4\pi} d_{\ell_p \ell_h \ell_1} R_{\ell_p}(r_1) R_{\ell_h}(r_1)
\end{aligned}$$

and the exchange overlap integrals are

$$\begin{aligned}
O^E_{t_1 s_1 \ell_1 k \ell_t, \ell_a \ell_b} &= J\int dr_b \int dr_a r_b r_a \chi_{\ell_b}(r_b) f^E_{t_1 s_1 \ell_1 k \ell_t, \ell_b \ell_a}(r_b, r_a)\chi_{\ell_a}(r_a) \\
f^E_{t_1 s_1 \ell_1 k \ell_t, \ell_b \ell_a}(r_b, r_a) &= J\ 4\pi\ m_a^k \sum_{\lambda_a \lambda_b \ell_\alpha \ell_\beta} [\frac{(2k+1)!}{(2\lambda_a+1)!(2\lambda_b+1)!}]^{1/2}\delta_{\lambda_a+\lambda_b,k}\ (-r_a)^{\lambda_a}(r_b)^{\lambda_b} \\
&\quad \times\ X(\ell_\alpha \lambda_a \ell_a, \ell_\beta \lambda_b \ell_b; \ell_1 k \ell_t) d_{\ell_\alpha \lambda_a \ell_a} d_{\ell_\beta \lambda_b \ell_b}\ c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a) \\
c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a) &= \frac{2\pi}{\hat{\ell}_1^2}\sum_{m_{\ell_1}} \hat{\ell}_\beta(\ell_\alpha m_{\ell_1} \ell_\beta 0|\ell_1 m_{\ell_1})\sum_{\ell\lambda}\hat{\ell}(\ell 0 \lambda m_{\ell_1}|\ell_1 m_{\ell_1}) \\
&\quad \times\ \int d\mu G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) Y_{\lambda m_{\ell_1}}(\theta', \pi) Y^*_{\ell_\alpha m_{\ell_1}}(\theta, 0) \\
G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) &= \frac{1}{\sqrt{4\pi}} r^{-k} V^E_{t_1 s_1 k}(r)\sum_{\lambda_1 \lambda_2 \ell_c}(-)^{\ell}\hat{\lambda}_1\hat{\lambda}_2(\lambda_1 0 \lambda_2 0|\lambda 0) W(\lambda_1 \lambda_2 \ell_1 \ell; \lambda \ell_c) \\
&\quad \times\ \int r_1^2 dr_1 \rho^E_{P,\lambda_2 \ell \ell_c}(r_b, r_1, r)\rho^E_{T,\ell_1 \lambda_1 \ell_c}(r_1, r) \\
\rho^E_{P,\lambda_2 \ell \ell_c}(r_b, r_1, r) &= \frac{2\pi}{\hat{\lambda}_2^2}\sum_m \hat{\ell}(\ell 0 \ell_c m|\lambda_2 m)\int \rho^E_{P,\lambda_2 \ell \ell_c}(r_b, r_1, \mu, r) Y_{\lambda_2 m}(\theta_2', 0) Y^*_{\ell_c m}(\theta, 0) d\mu \\
\rho^E_{T,\ell_1 \lambda_1 \ell_c}(r_1, r) &= \sum_{ph,\eta_1} i^{\ell_p+\ell_h-\pi} X(\ell_p \tfrac{1}{2} j_p, \ell_h \tfrac{1}{2} j_h; \ell_1 s_1 j_t)\ < I_B||[\hat{a}^\dagger_{j_p \nu_p}\hat{a}_{j_h \tilde{\nu}_h}]_{j_t}||I_A > R_{\ell_p}(r_1) \\
&\quad \times\ (-)^{\eta_1}\hat{\ell}_h \hat{\ell}_c \hat{\eta}_1(\ell_c 0 \eta_1 0|\ell_p 0)\ W(\ell_c \eta_1 \ell_1 \ell_h; \ell_p \lambda_1) \\
&\quad \times\ \frac{2\pi}{\hat{\ell}_h^2}\sum_{m_1}\hat{\eta}_1(\eta_1 0 \lambda_1 m_1|\ell_h m_1)\int R_{\ell_h}(r_1') Y_{\ell_h m_1}(\theta, 0) Y^*_{\lambda_1 m_1}(\theta', 0) d\mu'
\end{aligned}$$

2 Form Factors

2.1 Direct form factor

In Section I-10, the radial direct form factor gives an 8-dimensional integral, [Eq.(22)],

$$f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) \;=\; i^{-\pi}\int d\hat{r}_a \int d\vec{r}_1 \int d\vec{r}\; Y^*_{\ell_t m_{\ell_t}}(\hat{r}_a)\rho^D_P(\vec{r}_2)V^D_{t_1 s_1 k}(r)[Y_k(\hat{r})\rho^D_{T,\ell_1}(\vec{r}_1)]_{\ell_t m_{\ell_t}}$$

1) Projectile density

We transform $\rho^D_P(\vec{r}_2)$ into a function of $(\vec{r},\vec{r}_1,\vec{r}_a)$, as shown in Fig.3,

$$\rho^D_P(\vec{r}_2) \;\Rightarrow\; \rho^D_P(\vec{r}\,'_2,\vec{r}) \;\Rightarrow\; \rho^D_P(\vec{r},\vec{r}_1,\vec{r}_a)$$

where $\vec{r}_2 = \vec{r}\,'_2 + \vec{r},\; \vec{r}\,'_2 = \vec{r}_1 - \vec{r}_a$.

The first step. $[\rho^D_P(\vec{r}_2) \;\Rightarrow\; \rho^D_P(\vec{r}\,'_2,\vec{r})$, where $\rho^D_P(\vec{r}_2)$ is assumed to be a scalar function of r_2.] (See Fig.3.)

$$\begin{aligned}
\rho^D_P(r_2) &= \sum_{\lambda_2}\rho^D_{P,\lambda_2}(r'_2,r)(-)^{\lambda_2}[Y_{\lambda_2}Y_{\lambda_2}]_{00} \quad \text{[Eq.(23)]}\\
\rho^D_{P,\lambda_2}(r'_2,r) &= \sqrt{\pi}\int_{-1}^{1}\rho^D_{P,\lambda_2}(r'_2,r,\mu)Y_{\lambda_2 0}(\theta,0)d\mu, \quad \text{[Eq.(25)]}\\
r_2^2 &= (r'_2)^2 + r^2 + 2r'_2 r\mu \qquad \mu\equiv\cos\theta = \hat{r}\cdot\hat{r}'_2
\end{aligned}$$

See A-5 for the proof. The SUBROUTINE PDENST(0) calculates $\rho^D_{P,\lambda_2}(r'_2,r)$ and stores as RHOD(r'_2,r,λ_2).

The second step. $[\rho^D_{P,\lambda_2}(r'_2,r)Y_{\lambda_2\mu_2}(\hat{r}'_2) \;\Rightarrow\; \rho^D_{P,\ell\lambda_1}(r_a,r_1,r)Y_{\ell_a m_a}(\hat{r}_a)Y_{\ell m}(\hat{r}_1)]$ (See Fig.4.)

$$\begin{aligned}
\rho^D_{P,\lambda_2}(r'_2,r)Y_{\lambda_2\mu_2}(\hat{r}'_2) &= \sqrt{4\pi}\sum_{\ell\ell_q}\rho^D_{P,\lambda_2\ell\ell_q}(r_a,r_1,r)[Y_\ell(\hat{r}_a)Y_{\ell_q}(\hat{r}_1)]_{\lambda_2\mu_2} \quad \text{[Eq.(24)]}\\
\rho^D_{P,\lambda_2\ell\ell_q}(r_a,r_1,r) &= \frac{2\pi}{\hat{\lambda}_2^2}\sum_m \hat{\ell}(\ell 0\ell_q m|\lambda_2 m)\int \rho^D_{P,\lambda_2\ell\ell_q}(r_a,r_1,\mu,r)Y_{\lambda_2 m}(\theta'_2,0)Y^*_{\ell_q m}(\theta,0)d\mu\\
(r'_2)^2 &= r_1^2 + r_a^2 - 2r_1 r_a\mu \qquad \mu\equiv\cos\theta = \hat{r}_a\cdot\hat{r}_1,\\
&\qquad\qquad \mu'\equiv\cos\theta' = \hat{r}_a\cdot\hat{r}'_2 = \frac{r_1\mu - r_a}{r'_2} \quad \text{[Eq.(26)]}
\end{aligned}$$

See Appendix C.2 for the proof.

2) Angular integrations

$$\begin{aligned}
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi}\int d\hat{r}_a\int d\vec{r}_1\int d\vec{r}\; Y^*_{\ell_t m_{\ell_t}}(\hat{r}_a)\sum_{\lambda_2}\rho^D_{P,\lambda_2}(r'_2,r)(-)^{\lambda_2}[Y_{\lambda_2}Y_{\lambda_2}]_{00}\\
&\qquad \times\; V^D_{t_1 s_1 k}(r)[Y_k(\hat{r})\rho^D_{T,\ell_1}(\vec{r}_1)]_{\ell_t m_{\ell_t}}
\end{aligned}$$

Now we have relations

$$\begin{aligned}
[Y_{\lambda_2}Y_{\lambda_2}]_{00} &= \sum_{\mu_2}(\lambda_2\mu_2\lambda_2,-\mu_2|00)Y^*_{\lambda_2\mu_2}(\hat{r}'_2)Y_{\lambda_2\mu_2}(\hat{r})\\
&= \sum_{\mu_2}(-)^{\lambda_2}\hat{\lambda}_2^{-1}Y^*_{\lambda_2\mu_2}(\hat{r}'_2)Y_{\lambda_2\mu_2}(\hat{r})
\end{aligned}$$

$$
\begin{aligned}
\rho^D_{P,\lambda_2}(r_2',r)Y_{\lambda_2\mu_2}(\hat{r}_2') &= \sqrt{4\pi}\sum_{\ell\ell_q}\rho^D_{P,\lambda_2\ell\ell_q}(r_a,r_1,r)[Y_\ell(\hat{r}_a)Y_{\ell_q}(\hat{r}_1)]_{\lambda_2\mu_2} \\
\rho^D_{T,\ell_1 m_{\ell_1}}(\vec{r}_1) &= \frac{1}{\sqrt{4\pi}}\rho^D_{T,\ell_1}(r_1)Y_{\ell_1 m_{\ell_1}}(\hat{r}_1) \\
[Y_k(\hat{r})\rho^D_{T,\ell_1}(\vec{r}_1)]_{\ell_t m_{\ell_t}} &= \frac{1}{\sqrt{4\pi}}\rho^D_{T,\ell_1}(r_1)\sum_{qm_{\ell_1}}(kq\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})Y^*_{kq}(\hat{r})Y_{\ell_1 m_{\ell_1}}(\hat{r}_1)
\end{aligned}
$$

Thus the radial form factor, [Eq.(27)], becomes

$$
\begin{aligned}
f^D_{t_1 s_1\ell_1 k\ell_t}(r_a) &= i^{-\pi}\int r^2 dr V^D_{t_1 s_1 k}(r)\int r_1^2 dr_1\sum_{\ell\ell_q}\rho^D_{P,\lambda_2\ell\ell_q}(r_a,r_1,r)\rho^D_{T,\ell_1}(r_1) \\
&\times\sum_{\text{all } m}(-)^{\lambda_2}(-)^{\lambda_2}\hat{\lambda}_2^{-1}\frac{1}{\sqrt{4\pi}}\sqrt{4\pi}(kq\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})(\ell_a 0\ell m|\lambda_2 m) \\
&\times\int d\hat{r}_a\; Y^*_{\ell_t m_{\ell_t}}(\hat{r}_a)Y_{\ell m}(\hat{r}_a)\int d\hat{r}\; Y^*_{kq}(\hat{r})Y_{\lambda_2\mu_2}(\hat{r})\int d\hat{r}_1\; Y^*_{\ell_q 0}(\hat{r}_1)Y_{\ell_1 m_{\ell_1}}(\hat{r}_1) \\
&= i^{-\pi}\int r^2 dr V^D_{t_1 s_1 k}(r)\int r_1^2 dr_1\rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r)\rho^D_{T,\ell_1}(r_1) \\
&\times\sum_{qm_{\ell_1}}\hat{\lambda}_2^{-1}(kq\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})(\ell_1 m_{\ell_1}\ell_t m_{\ell_t}|\lambda_2\mu_2) \\
&= i^{-\pi}(-)^{\ell_1}\hat{\ell}_t^{-1}\int r^2 dr V^D_{t_1 s_1 k}(r)\int r_1^2 dr_1\rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r)\rho^D_{T,\ell_1}(r_1)
\end{aligned}
$$

where we use $(\ell_1 m_{\ell_1}\ell_t m_{\ell_t}|\lambda_2\mu_2) = (-)^{\ell_1}\hat{\lambda}_2\hat{\ell}_t^{-1}(\lambda_2\mu_2\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})$. Note that the angular integrations give $\lambda_2 = k, \ell = \ell_t, \ell_q = \ell_1$ and also $\vec{\ell}_1 + \vec{k} = \vec{\ell}_t$.

We now summarize the radial direct form factors as

$$
\begin{aligned}
f^D_{t_1 s_1\ell_1 k\ell_t}(r_a) &= i^{-\pi}(-)^{\ell_1}\hat{\ell}_t^{-1}\int r^2 dr V^D_{t_1 s_1 k}(r)\int r_1^2 dr_1\rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r)\rho^D_{T,\ell_1}(r_1) \\
\rho^D_{P,k\ell_t\ell_1}(r_a,r_1,r) &= \frac{2\pi}{\hat{k}^2}\sum_m\hat{\ell}_t(\ell_t 0\ell_1 m|km)\int\rho^D_{P,k}(r_a,r_1,\mu,r)Y_{km}(\theta_2',0)Y^*_{\ell_1 m}(\theta,0)d\mu \\
\rho^D_{T,\ell_1}(r_1) &= X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1 s_1 j_t)\; < I_B||[\hat{a}^\dagger_{j_p\nu_p}\hat{a}_{\tilde{j_h\nu_h}}]_{j_t}||I_A > \sqrt{4\pi}d_{\ell_p\ell_h\ell_1}R_{\ell_p}(r_1)R_{\ell_h}(r_1)
\end{aligned}
$$

2.2 Exchange form factor

In Section 1.10, the radial exchange form factor gives an 7-dimensional integral, [Eq.(28)],

$$\begin{aligned} f^E_{t_1 s_1 \ell_1 k \ell_t}(r_b, r_a) &= J i^{-\pi} \int d\hat{r}_a \int d\hat{r}_b \int d\vec{r}_1 \; [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^*_{\ell_t m_{\ell_t}} \\ &\times \; \rho^E_P(\vec{r}_2, \vec{r}_2') V^E_{t_1 s_1 k}(r) [\rho^E_{T,\ell_1}(\vec{r}_1, \vec{r}_1') Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \end{aligned}$$

1) Projectile density

We first expand the projectile scalar density function $\rho^E_P(\vec{r}_2, \vec{r}_2')$ in the multipoles (See Appendix C.2.) in order to have a function of $(\vec{r}, \vec{r}_2')$ as seen Fig.5,

$$\begin{aligned} \rho^E_P(\vec{r}_2, \vec{r}_2') &= \sum_{\ell_2} [\phi_{\ell_2}(\hat{r}_2') \phi_{\tilde{\ell}_2}(\hat{r}_2)]_{00} \\ &= \sum_{\ell_2 m_2} (\ell_2 m_2 \ell_2, -m_2 | 00) \omega_{\ell_2}(r_2') Y_{\ell_2 m_2}(\hat{r}_2') \omega_{\ell_2}(r_2) Y_{\ell_2 m_2}(\hat{r}_2) \\ &= \sum_{\ell_2 m_2} \hat{\ell}_2^{-1} \omega_{\ell_2}(r_2') Y_{\ell_2 m_2}(\hat{r}_2') \sqrt{4\pi} \sum_{\lambda_2 \eta_2} [Y_{\eta_2}(\hat{r}_2') Y_{\lambda_2}(\hat{r})]_{\ell_2 m_2} \\ &\times \; \frac{2\pi}{2\ell_2 + 1} \sum_m \hat{\eta}_2 (\eta_2 0 \lambda_2 m | \ell_2 m) \int \omega_{\ell_2}(r_2) Y_{\ell_2 m}(\theta, 0) Y^*_{\lambda_2 m}(\theta', 0) d\mu \\ (r_2)^2 &= (r_1')^2 + r^2 - 2 r_2' r \mu \qquad \mu \equiv \cos\theta = \hat{r} \cdot r_2', \\ & \qquad\qquad\qquad\qquad \mu' \equiv \cos\theta' = \hat{r} \cdot \hat{r}_2 = \frac{r_2' \mu + r}{r_2} \end{aligned}$$

The spherical harmonics can be coupled,

$$\begin{aligned} Y's &= \sum_{\ell_2 m_2, \lambda_2 \eta_2} Y_{\ell_2 m_2}(\hat{r}_2') [Y_{\eta_2}(\hat{r}_2') Y_{\lambda_2}(\hat{r})]_{\ell_2 m_2} \\ &= \sum_{\ell_2 m_2, \lambda_2 \mu_2 \eta_2 \nu_2} (\eta_2 \nu_2 \lambda_2 \mu_2 | \ell_2 m_2) Y_{\ell_2 m_2}(\hat{r}_2') Y_{\eta_2 \nu_2}(\hat{r}_2') Y_{\lambda_2 \mu_2}(\hat{r}) \\ &= \sum (\eta_2 \nu_2 \lambda_2 \mu_2 | \ell_2 m_2) \sum_{\lambda_1} \frac{\hat{\ell}_2 \hat{\eta}_2}{\hat{\lambda}_1 \sqrt{4\pi}} (\eta_2 0 \ell_2 0 | \lambda_1 0) (\eta_2 \nu_2 \ell_2 m_2 | \lambda_1 \mu_1) Y_{\lambda_1 \mu_1}(\hat{r}_2') Y_{\lambda_2 \mu_2}(\hat{r}) \\ &= \sum_{\ell_2 \lambda_2 \eta_2} \frac{\hat{\ell}_2 \hat{\eta}_2}{\hat{\lambda}_2 \sqrt{4\pi}} (\eta_2 0 \lambda_2 0 | \ell_2 0) (-)^{\lambda_2} \hat{\lambda}_2 [Y_{\lambda_2}(\hat{r}_2') Y_{\lambda_2}(\hat{r})]_{00} \delta(\lambda_1, \lambda_2) \\ &\quad \times \sum_{m_2 \nu_2} (\eta_2 \nu_2 \ell_2 m_2 | \lambda_2 \mu_2) (\eta_2 \nu_2 \ell_2 m_2 | \lambda_1, \mu_1) \delta(\mu_1, -\mu_2) \end{aligned}$$

We finally obtain the nonlocal projectile density, [Eq.(30a)][Eq.(31a)],

$$\begin{aligned} \rho^E_P(\vec{r}_2, \vec{r}_2') &= \sum_{\lambda_2} \rho^E_{P,\lambda_2}(r_2', r) (-)^{\lambda_2} [Y_{\lambda_2}(\hat{r}_2') Y_{\lambda_2}(\hat{r})]_{00} \\ \rho^E_{P,\lambda_2}(r_2', r) &= \sum_{\ell_2 \eta_2} \hat{\ell}_2 \hat{\eta}_2 \omega_{\ell_2}(r_2') (\eta_2 0 \lambda_2 0 | \ell_2 0) \\ &\quad \times \; \frac{2\pi}{2\ell_2 + 1} \sum_m \hat{\eta}_2 (\eta_2 0 \lambda_2 m | \ell_2 m) \int \omega_{\ell_2}(r_2) Y_{\ell_2 m}(\theta, 0) Y^*_{\lambda_2 m}(\theta', 0) d\mu \end{aligned}$$

This $\rho^E_{P,\lambda_2}(r_2', r)$ is stored as RHOE(N2P,NH,LAM2P1) in the SUBROUTINE PDENST(1).

We have done the first step in the change of coordinates, $\rho^E_P(\vec{r}_2, \vec{r}_2') \to \rho^E_P(\vec{r}_2', \vec{r}) \to \rho^E_P(\vec{r}_b, \vec{r}_1, \vec{r})$, and now do the second step. This can be written in exactly the same way as done in the direct case. (See Fig.6.)

$$\begin{aligned}\rho^E_P(\vec r_2,\vec r_2') &= \sum_{\lambda_2}\rho^E_{P,\lambda_2}(r_2',r)(-)^{\lambda_2}[Y_{\lambda_2}(\hat r_2')Y_{\lambda_2}(\hat r)]_{00}\\ &= \sum_{\lambda_2\mu_2}\rho^E_{P,\lambda_2}(r_2',r)Y_{\lambda_2\mu_2}(\hat r_2')\hat\lambda_2^{-1}Y_{\lambda_2\mu_2}(\hat r)\end{aligned}$$

$$\begin{aligned}\rho^E_{P,\lambda_2}(r_2',r)Y_{\lambda_2\mu_2}(\hat r_2') &= \sqrt{4\pi}\sum_{\ell\ell_q}\rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,r)[Y_{\ell m}(\hat r_b)Y_{\ell_q m_q}(\hat r_1)]_{\lambda_2\mu_2}\quad \text{[Eq.(32)]}\\ \rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,r) &= \frac{2\pi}{\hat\lambda_2^2}\sum_m \hat\ell(\ell 0\ell_q m|\lambda_2 m)\int\rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,\mu,r)Y_{\lambda_2 m}(\theta_2',0)Y^*_{\ell_q m}(\theta,0)d\mu\\ (r_2')^2 &= r_1^2+r_b^2-2r_1r_b\mu\qquad \mu\equiv\cos\theta=\hat r_b\cdot r_1,\\ &\qquad\qquad \mu'\equiv\cos\theta'=\hat r_b\cdot r_2'=\frac{r_1\mu-r_b}{r_2'}\end{aligned}$$

2) Target density

The exchange target non-local density becomes (See Section 1.5.)

$$\begin{aligned}\rho^E_{T,\ell_1 m_{\ell_1}}(\vec r_1,\vec r_1') &= \sum_{ph}X(\ell_p\tfrac12 j_p,\ell_h\tfrac12 j_h;\ell_1 s_1 j_t)\ < I_B||[\hat a^\dagger_{j_p\nu_p}\hat a_{\tilde{j_h\nu_h}}]_{j_t}||I_A>[\phi_{\ell_p}(\vec r_1)\phi^*_{\tilde\ell_h}(\vec r_1')]_{\ell_1 m_{\ell_1}}\\ &= \sum_{ph}X(\ell_p\tfrac12 j_p,\ell_h\tfrac12 j_h;\ell_1 s_1 j_t)\ < I_B||[\hat a^\dagger_{j_p\nu_p}\hat a_{\tilde{j_h\nu_h}}]_{j_t}||I_A>\\ &\qquad i^{\ell_p+\ell_h-\pi}R_{\ell_p}(r_1)R_{\ell_h}(r_1')(\ell_p m_p\ell_h,-m_h|\ell_1 m_{\ell_1})Y_{\ell_p m_p}(\hat r_1)Y^*_{\ell_h,-m_h}(\hat r_1')\end{aligned}$$

We now change coordinates such that $\rho^E_T(\vec r_1,\vec r_1')\to\rho^E_T(\vec r_1,\vec r)$ (See Fig. 7.) and we have

$$\begin{aligned}R_{\ell_h}(r_1')Y^*_{\ell_h,-m_h}(\hat r_1') &= \sqrt{4\pi}\sum_{\eta_1\lambda_1}R_{\ell_h\eta_1\lambda_1}(r_1,r)[Y_{\eta_1}Y_{\lambda_1}]_{\ell_h m_h}\\ &= \sqrt{4\pi}\sum_{\eta_1\lambda_1}R_{\ell_h\eta_1\lambda_1}(r_1,r)\sum_{\nu_1\mu_1}(\eta_1\nu_1\lambda_1\mu_1|\ell_h m_h)Y_{\eta_1\nu_1}(\hat r_1)Y_{\lambda_1\mu_1}(\hat r)\\ R_{\ell_h\eta_1\lambda_1}(r_1,r) &= \frac{2\pi}{2\ell_h+1}\sum_{m_1}\hat\eta_1(\eta_1 0\lambda_1 m_1|\ell_h m_1)\int R_{\ell_h}(r_1')Y_{\ell_h m_1}(\theta,0)Y^*_{\lambda_1 m_1}(\theta',0)d\mu'\\ (r_1')^2 &= (r_1)^2+r^2+2r_1r\mu\qquad \mu\equiv\cos\theta=\hat r_1\cdot r,\\ &\qquad\qquad \mu'\equiv\cos\theta'=\hat r_1\cdot\hat r_1'=\frac{r\mu+r_1}{r_1'}\\ Y_{\ell_p m_p}(\hat r_1)Y_{\eta_1\nu_1}(\hat r_1) &= \sum_{\ell_c}\frac{\hat\ell_p\hat\eta_1}{\sqrt{4\pi}\hat\ell_c}(\ell_p 0\eta_1 0|\ell_c 0)(\ell_p m_p\eta_1\nu_1|\ell_c m_c)Y_{\ell_c m_c}(\hat r_1)\\ Y_{\ell_c m_c}(\hat r_1)Y_{\lambda_1\mu_1}(\hat r) &= \sum_{\ell_1 m_1}(\ell_c m_c\lambda_1\mu_1|\ell_1 m_1)[Y_{\ell_c}(\hat r_1)Y_{\lambda_1}(\hat r)]_{\ell_1 m_1}\end{aligned}$$

Combining CG's gives

$$\begin{aligned}\text{Geometry} &= \sqrt{4\pi}\frac{\hat\ell_p\hat\eta_1}{\sqrt{4\pi}\hat\ell_c}(\ell_p 0\eta_1 0|\ell_c 0)\\ &\quad\sum_{all\ m}(\ell_p m_p\ell_h,-m_h|\ell_1 m_{\ell_1})(\eta_1\nu_1\lambda_1\mu_1|\ell_h m_h)(\ell_p m_p\eta_1\nu_1|\ell_c m_c)(\ell_c m_c\lambda_1\mu_1|\ell_1 m_1)\\ &= (-)^{\eta_1}\hat\ell_h\hat\ell_c\hat\eta_1(\ell_c 0\eta_1 0|\ell_p 0)\ W(\ell_c\eta_1\ell_1\ell_h;\ell_p\lambda_1)\end{aligned}$$

Finally we obtain the non-local target density, [Eq.(30b)] and [Eq.(31b)],

$$
\begin{aligned}
\rho^E_{T,\ell_1 m_{\ell_1}}(\vec{r}_1,\vec{r}_1') &= \sum_{\lambda_1\ell_c} \rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r)[Y_{\ell_c}(\hat{r}_1)Y_{\lambda_1}(\hat{r})]_{\ell_1 m_{\ell_1}} \\
\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r) &= \sum_{ph,\eta_1} i^{\ell_p+\ell_h-\pi} X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1 s_1 j_t)\ < I_B||[\hat{a}^\dagger_{j_p\nu_p}\hat{a}_{j_h\tilde{\nu}_h}]_{j_t}||I_A > R_{\ell_p}(r_1) \\
&\quad \times\ (-)^{\eta_1}\hat{\ell}_h\hat{\ell}_c\hat{\eta}_1(\ell_c 0\eta_1 0|\ell_p 0)\ W(\ell_c\eta_1\ell_1\ell_h;\ell_p\lambda_1) \\
&\quad \times\ \frac{2\pi}{2\ell_h+1}\sum_{m_1}\hat{\eta}_1(\eta_1 0\lambda_1 m_1|\ell_h m_1)\int R_{\ell_h}(r_1')Y_{\ell_h m_1}(\theta,0)Y^*_{\lambda_1 m_1}(\theta',0)d\mu'
\end{aligned}
$$

The last line is defined as GWT(N1,NH,NGW) (NGW=λ_1,η_1) in the SUBROUTINE DENST. The term $\rho^E_{T,\lambda_1\ell_c}(r_1,r)$ is stored as TRHO(N1,NH,KB) (KB=$(t_1 s_1\ell_1),\lambda_1,\ell_c$) in the SUBROUTINE TRECAL.

3) Angular integration of $\vec{r}_1$

We define $c_{t_1 s_1\ell_1 m_{\ell_1}}(\vec{r}_b,\vec{r})$, [Eq.(29)], as

$$
\begin{aligned}
c_{t_1 s_1\ell_1 m_{\ell_1}}(\vec{r}_b,\vec{r}) &\equiv \frac{1}{\sqrt{4\pi}}r^{-k}V^E_{t_1 s_1 k}(r)\int d\vec{r}_1\rho^E_P(\vec{r}_2,\vec{r}_2')\rho^E_{T,\ell_1 m_{\ell_1}}(\vec{r}_1,\vec{r}_1') \\
&= r^{-k}V^E_{t_1 s_1 k}(r)\int d\vec{r}_1\sum_{\lambda_2\mu_2\ell\ell_q}\rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,r)[Y_\ell(\hat{r}_b)Y_{\ell_q}(\hat{r}_1)]_{\lambda_2\mu_2} \\
&\qquad \times\ \hat{\lambda}_2^{-1}Y_{\lambda_2\mu_2}(\hat{r})\ \sum_{\lambda_1\ell_c}\rho^E_{T,\lambda_1\ell_c}(r_1,r)[Y_{\ell_c}(\hat{r}_1)Y_{\lambda_1}(\hat{r})]_{\ell_1 m_{\ell_1}} \\
&= r^{-k}V^E_{t_1 s_1 k}(r)\sum_{\lambda_2\mu_2\ell\ell_q\lambda_1\ell_c}\int d\hat{r}_1\ (\mathrm{AI})\ \int r_1^2 dr_1\rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,r)\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\mu_2}\int d\hat{r}_1\ (\mathrm{AI}) &= \sum_{\mu_2}\int d\hat{r}_1\ [Y_\ell(\hat{r}_b)Y_{\ell_q}(\hat{r}_1)]_{\lambda_2\mu_2}\hat{\lambda}_2^{-1}Y_{\lambda_2\mu_2}(\hat{r})[Y_{\ell_c}(\hat{r}_1)Y_{\lambda_1}(\hat{r})]_{\ell_1 m_{\ell_1}} \\
&= \int d\hat{r}_1\ \hat{\lambda}_2^{-1}\sum_{m_{\ell_q}m m_c\mu_1\mu_2}(\ell m\ell_q m_{\ell_q}|\lambda_2\mu_2)Y_{\ell m}(\hat{r}_b)Y_{\ell_q m_{\ell_q}}(\hat{r}_1) \\
&\qquad (\ell_c m_c\lambda_1\mu_1|\ell_1 m_{\ell_1})Y_{\ell_c m_c}(\hat{r}_1)Y_{\lambda_1\mu_1}(\hat{r})Y_{\lambda_2\mu_2}(\hat{r}) \\
&= \hat{\lambda}_2^{-1}\sum_{m_{\ell_q}m m_c\mu_1\mu_2}(\ell m\ell_q m_{\ell_q}|\lambda_2\mu_2)(\ell_c m_c\lambda_1\mu_1|\ell_1 m_{\ell_1})\int d\hat{r}_1\ Y_{\ell_q m_{\ell_q}}(\hat{r}_1)Y_{\ell_c m_c}(\hat{r}_1) \\
&\qquad \times\ \sum_\lambda\frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}}(\lambda_1 0\lambda_2 0|\lambda 0)(\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)Y_{\lambda\mu}(\hat{r})Y_{\ell m}(\hat{r}_b) \\
&= \sum_\lambda\frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}}(\lambda_1 0\lambda_2 0|\lambda 0)\sum_{m m_c\mu_1\mu_2}(\ell m\ell_c m_c|\lambda_2\mu_2)(\ell_c m_c\lambda_1\mu_1|\ell_1 m_{\ell_1}) \\
&\qquad (\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)(\lambda\mu\ell_p m_{\ell_p}|\ell_1 m_{\ell_1})[Y_\lambda(\hat{r})Y_{\ell_p}(\hat{r}_b)]_{\ell_1 m_{\ell_1}} \\
&= \sum_\lambda\frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}}(\lambda_1 0\lambda_2 0|\lambda 0)(-)^\ell W(\lambda_1\lambda_2\ell_1\ell;\lambda\ell_c)[Y_\lambda(\hat{r})Y_{\ell_b}(\hat{r}_b)]_{\ell_1 m_{\ell_1}}
\end{aligned}
$$

Note that this angular integration gives $\ell_q=\ell_c$. We thus obtain [Eq.(33a)] and [Eq.(33b)],

$$
c_{t_1 s_1\ell_1 m_{\ell_1}}(\vec{r}_b,\vec{r}) = \sum_{\ell_p\lambda} i^\pi G^k_{t_1 s_1\ell_1\ell\lambda}(r_b,r)[Y_\lambda(\hat{r})Y_\ell(\hat{r}_b)]_{\ell_1 m_{\ell_1}}
$$

$$G^{k}_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) = \frac{1}{\sqrt{4\pi}} r^{-k} V^{E}_{t_1 s_1 k}(r) \sum_{\lambda_1 \lambda_2 \ell_c} (-)^{\ell_p} \hat{\lambda}_1 \hat{\lambda}_2 (\lambda_1 0 \lambda_2 0 | \lambda 0) W(\lambda_1 \lambda_2 \ell_1 \ell; \lambda \ell_c)$$
$$\times \int r_1^2 dr_1 \rho^{E}_{P,\lambda_2 \ell \ell_c}(r_b, r_1, r) \rho^{E}_{T,\ell_1 \lambda_1 \ell_c}(r_1, r)$$

The radial integration in the above equation is stored as QA(KC,NH) (KC=$\lambda_1\lambda_2\ell\ell_c$), and G factor as GGRI(KA,NH,NLS) (KA=$\ell\lambda$) in the SUBROUTINE FFCALE.

4) Integrand of $\hat{r}_a$ and $\hat{r}_b$

We are now ready to transform $c_{t_1 s_1 \ell_1 m_{\ell_1}}(\vec{r}_b, \vec{r})$ to $c_{t_1 s_1 \ell_1 m_{\ell_1}}(\vec{r}_b, \vec{r}_a)$,

$$c_{t_1 s_1 \ell_1 m_{\ell_1}}(\vec{r}_b, \vec{r}_a) = \sum_{\ell_\alpha \ell_\beta} c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a) [Y_{\ell_\alpha}(\hat{r}_a) Y_{\ell_\beta}(\hat{r}_b)]_{\ell_1 m_{\ell_1}}$$

where the expansion coefficient is calculated as (See Fig. 8.)

$$c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a) = \frac{2\pi}{\hat{\ell}_1^2} \sum_{m_{\ell_1}} \hat{\ell}_\beta (\ell_\alpha m_{\ell_1} \ell_\beta 0 | \ell_1 m_{\ell_1}) \sum_{\ell \lambda} \hat{\ell} (\ell 0 \lambda m_{\ell_1} | \ell_1 m_{\ell_1})$$
$$\times \int d\mu G^{k}_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) Y_{\lambda m_{\ell_1}}(\theta', \pi) Y^{*}_{\ell_\alpha m_{\ell_1}}(\theta, 0)$$
$$(r/a)^2 = r_b^2 + r_a^2 - 2 r_b r_a \mu \qquad \mu \equiv \cos\theta = \hat{r}_b \cdot r_a,$$
$$\mu' \equiv \cos\theta' = \hat{r}_b \cdot \hat{r} = \frac{r_a \mu - r_b}{r/a}$$

We note that ℓ_t should be read to ℓ_1 in the equation for $c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a)$, [Eq.(35)].

5) Angular integrations of $\vec{r}_a$ and $\vec{r}_b$

We now end up to obtain exchange form factors by integrating $\vec{r}_a$ and $\vec{r}_b$,

$$f^{E}_{t_1 s_1 \ell_1 k \ell_t, \ell_b \ell_a}(r_b, r_a) = J i^{-\pi} \int d\hat{r}_a \int d\hat{r}_b \int d\vec{r}_1 \, [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^{*}_{\ell_t m_{\ell_t}}$$
$$\times \rho^{E}_{P}(\vec{r}_2, \vec{r}_2') V^{E}_{t_1 s_1 k}(r) [\rho^{E}_{T,\ell_1}(\vec{r}_1, \vec{r}_1') Y_k(\hat{r})]_{\ell_t m_{\ell_t}}$$
$$= J \int d\hat{r}_a \int d\hat{r}_b \, [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^{*}_{\ell_t m_{\ell_t}} \sum_{m_{\ell_1} q} (\ell_1 m_{\ell_1} k q | \ell_t m_{\ell_t}) Y_{kq}(\hat{r})$$
$$\times \sqrt{4\pi} r^k \sum_{\ell_\alpha \ell_\beta} c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a) [Y_{\ell_\alpha}(\hat{r}_a) Y_{\ell_\beta}(\hat{r}_b)]_{\ell_1 m_{\ell_1}}$$
$$= J \, 4\pi \, m_a^k \sum_{\lambda_a \lambda_b \ell_\alpha \ell_\beta} [\frac{(2k+1)!}{(2\lambda_a+1)!(2\lambda_b+1)!}]^{1/2} \delta_{\lambda_a+\lambda_b,k} \, (-r_a)^{\lambda_a} (r_b)^{\lambda_b}$$
$$\times \int d\hat{r}_a \int d\hat{r}_b [Y_{\lambda_a}(\hat{r}_a) Y_{\lambda_b}(\hat{r}_b)]_{kq} \, [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^{*}_{\ell_t m_{\ell_t}} \sum_{m_{\ell_1} q} (\ell_1 m_{\ell_1} k q | \ell_t m_{\ell_t})$$
$$\times \, [Y_{\ell_\alpha}(\hat{r}_a) Y_{\ell_\beta}(\hat{r}_b)]_{\ell_1 m_{\ell_1}} c_{t_1 s_1 \ell_1 k, \ell_\alpha \ell_\beta}(r_b, r_a)$$

where the integrand of angular integrations becomes

$$(\mathrm{AI}) \equiv [Y_{\lambda_a}(\hat{r}_a) Y_{\lambda_b}(\hat{r}_b)]_{kq} \, [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^{*}_{\ell_t m_{\ell_t}} [Y_{\ell_\alpha}(\hat{r}_a) Y_{\ell_\beta}(\hat{r}_b)]_{\ell_1 m_{\ell_1}}$$
$$= \sum_{all\ m} (\lambda_a \mu_a \lambda_b \mu_b | k q)(\ell_\alpha m_{\ell_\alpha} \ell_\beta m_{\ell_\beta} | \ell_1 m_{\ell_1})(\ell_a m_{\ell_a} \ell_b m_{\ell_b} | \ell_t m_{\ell_t})$$
$$\times \, Y_{\ell_\alpha m_{\ell_\alpha}}(\hat{r}_a) Y_{\lambda_a \mu_a}(\hat{r}_a) Y^{*}_{\ell_a m_{\ell_a}}(\hat{r}_a) Y_{\ell_\beta m_{\ell_\beta}}(\hat{r}_b) Y_{\lambda_b \mu_b}(\hat{r}_b) Y^{*}_{\ell_b m_{\ell_b}}(\hat{r}_b)$$

and the integration over $d\hat{r}_a$ and $d\hat{r}_b$ and summing over $m_{\ell_1}q$ gives,

$$\begin{aligned}
\sum_{m_{\ell_1}q}\int d\hat{r}_a\int d\hat{r}_b\ (\text{AI}) &= \sum_{all\ m}(\lambda_a\mu_a\lambda_b\mu_b|kq)(\ell_\alpha m_{\ell_\alpha}\ell_\beta m_{\ell_\beta}|\ell_1 m_{\ell_1})(\ell_a m_{\ell_a}\ell_b m_{\ell_b}|\ell_t m_{\ell_t}) \\
&\quad\times\ (\ell_1 m_{\ell_1}kq|\ell_t m_{\ell_t})(\ell_\alpha m_{\ell_\alpha}\lambda_a\mu_a|\ell_a m_{\ell_a})(\ell_\beta m_{\ell_\beta}\lambda_b\mu_b|\ell_b m_{\ell_b}) \\
&\quad\times\ d_{\ell_\alpha\lambda_a\ell_a}d_{\ell_\beta\lambda_b\ell_b} \\
&= X(\ell_\alpha\lambda_a\ell_a,\ell_\beta\lambda_b\ell_b;\ell_1 k\ell_t)d_{\ell_\alpha\lambda_a\ell_a}d_{\ell_\beta\lambda_b\ell_b}
\end{aligned}$$

We finally obtain [Eq.(36)],

$$\begin{aligned}
f^E_{t_1s_1\ell_1k\ell_t,\ell_b\ell_a}(r_b,r_a) &= J\ 4\pi\ m_a^k\sum_{\lambda_a\lambda_b\ell_\alpha\ell_\beta}[\frac{(2k+1)!}{(2\lambda_a+1)!(2\lambda_b+1)!}]^{1/2}\delta_{\lambda_a+\lambda_b,k}\ (-r_a)^{\lambda_a}(r_b)^{\lambda_b} \\
&\quad\times\ X(\ell_\alpha\lambda_a\ell_a,\ell_\beta\lambda_b\ell_b;\ell_1k\ell_t)d_{\ell_\alpha\lambda_a\ell_a}d_{\ell_\beta\lambda_b\ell_b}\ c_{t_1s_1\ell_1k,\ell_\alpha\ell_\beta}(r_b,r_a)
\end{aligned}$$

For the central interaction ($k=0$), $f^E_{t_1s_1\ell_10\ell_1,\ell_b\ell_a}(r_b,r_a)$ is just nothing but $c_{t_1s_1\ell_10,\ell_a\ell_b}(r_b,r_a)$, since for $k=0$, $\lambda_a=\lambda_b=0$, and

$$\begin{aligned}
d_{\ell_\alpha0\ell_a} &= \frac{1}{\sqrt{4\pi}}\frac{\hat{\ell}_\alpha}{\hat{\ell}_a}(\ell_\alpha000|\ell_a)=\frac{1}{\sqrt{4\pi}}\delta_{\ell_a\ell_\alpha}\qquad d_{\ell_\beta0\ell_b}=\frac{1}{\sqrt{4\pi}}\delta_{\ell_b\ell_\beta} \\
X(\ell_\alpha0\ell_a,\ell_\beta0\ell_b;\ell_10\ell_t) &= U\begin{pmatrix}\ell_\alpha & 0 & \ell_a\\ \ell_\beta & 0 & \ell_b\\ \ell_1 & 0 & \ell_t\end{pmatrix}=\hat{\ell}_1\hat{\ell}_a\hat{\ell}_b(-)^\sigma U\begin{pmatrix}\ell_a & \ell_\alpha & 0\\ \ell_b & \ell_\beta & 0\\ \ell_t & \ell_1 & 0\end{pmatrix} \\
&= \hat{\ell}_1\hat{\ell}_a\hat{\ell}_b(-)^\sigma(-)^{\ell_t-\ell_a-\ell_b}\hat{\ell}_1^{-1}W(\ell_a\ell_\alpha\ell_b\ell_\beta;0\ell_t) \\
&= \hat{\ell}_a\hat{\ell}_b(-)^\sigma(-)^{\ell_t-\ell_a-\ell_b}(-)^{-\ell_t+\ell_a+\ell_b}\hat{\ell}_a^{-1}\hat{\ell}_b^{-1} \\
&= 1
\end{aligned}$$

where $\sigma=\ell_a+\ell_b+\ell_\alpha+\ell_\beta+\ell_1+\ell_t=$ even.

We now summarize the radial exchange form factors as

$$\begin{aligned}
f^E_{t_1s_1\ell_1k\ell_t,\ell_b\ell_a}(r_b,r_a) &= J\ 4\pi\ m_a^k\sum_{\lambda_a\lambda_b\ell_\alpha\ell_\beta}[\frac{(2k+1)!}{(2\lambda_a+1)!(2\lambda_b+1)!}]^{1/2}\delta_{\lambda_a+\lambda_b,k}\ (-r_a)^{\lambda_a}(r_b)^{\lambda_b} \\
&\quad\times\ X(\ell_\alpha\lambda_a\ell_a,\ell_\beta\lambda_b\ell_b;\ell_1k\ell_t)d_{\ell_\alpha\lambda_a\ell_a}d_{\ell_\beta\lambda_b\ell_b}\ c_{t_1s_1\ell_1k,\ell_\alpha\ell_\beta}(r_b,r_a) \\
c_{t_1s_1\ell_1k,\ell_\alpha\ell_\beta}(r_b,r_a) &= \frac{2\pi}{\hat{\ell}_1^2}\sum_{m_{\ell_1}}\hat{\ell}_\beta(\ell_\alpha m_{\ell_1}\ell_\beta0|\ell_1m_{\ell_1})\sum_{\ell\lambda}\hat{\ell}(\ell0\lambda m_{\ell_1}|\ell_1m_{\ell_1}) \\
&\quad\times\ \int d\mu G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)Y_{\lambda m_{\ell_1}}(\theta',\pi)Y^*_{\ell_\alpha m_{\ell_1}}(\theta,0) \\
G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r) &= \frac{1}{\sqrt{4\pi}}r^{-k}V^E_{t_1s_1k}(r)\sum_{\lambda_1\lambda_2\ell_c}(-)^\ell\hat{\lambda}_1\hat{\lambda}_2(\lambda_10\lambda_20|\lambda0)W(\lambda_1\lambda_2\ell_1\ell;\lambda\ell_c) \\
&\quad\times\ \int r_1^2dr_1\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r)\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r) \\
\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r) &= \frac{2\pi}{\hat{\lambda}_2^2}\sum_m\hat{\ell}(\ell0\ell_cm|\lambda_2m)\int\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,\mu,r)Y_{\lambda_2m}(\theta_2',0)Y^*_{\ell_cm}(\theta,0)d\mu \\
\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r) &= \sum_{ph,\eta_1}i^{\ell_p+\ell_h-\pi}X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1s_1j_t)\ <I_B||[\hat{a}^\dagger_{j_p\nu_p}\hat{a}_{j_h\tilde{\nu}_h}]_{j_t}||I_A>R_{\ell_p}(r_1) \\
&\quad\times\ (-)^{\eta_1}\hat{\ell}_h\hat{\ell}_c\hat{\eta}_1(\ell_c0\eta_10|\ell_p0)\ W(\ell_c\eta_1\ell_1\ell_h;\ell_p\lambda_1) \\
&\quad\times\ \frac{2\pi}{\hat{\ell}_h^2}\sum_{m_1}\hat{\eta}_1(\eta_10\lambda_1m_1|\ell_hm_1)\int R_{\ell_h}(r_1')Y_{\ell_hm_1}(\theta,0)Y^*_{\lambda_1m_1}(\theta',0)d\mu'
\end{aligned}$$

3 Limiting Cases

3.1 Nucleon-nucleus scattering

1) Direct form factor

For the nucleon-nucleus scattering, we adopt the following limits for the direct form factor

$$
\begin{aligned}
\rho_P(\vec{r}_2) &= \delta(\vec{r}_2) = \sum_{\lambda_2} \rho^D_{P,\lambda_2}(r'_2, r)(-)^{\lambda_2}[Y_{\lambda_2}Y_{\lambda_2}]_{00} \\
\rho_{P,\lambda_2}(r'_2, r) &= \hat{\lambda}_2(-)^{\lambda_2}\delta(r'_2 - r)/r^2
\end{aligned}
$$

The direct form factor of Section 1.9 gives [Eq.(38a)],

$$
\begin{aligned}
F^D_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a) &= \int d\vec{r}_1 \int d\vec{r}_2 \rho^D_P(\vec{r}_2) V^D_{t_1 s_1 k}(r)[\rho^D_{T,\ell_1}(\vec{r}_1)Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \\
&= \int d\vec{r}_1 \int d\vec{r}_2 \delta(\vec{r}_2) V^D_{t_1 s_1 k}(r)[\rho^D_{T,\ell_1}(\vec{r}_1)Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \\
&= \int d\vec{r}_1 V^D_{t_1 s_1 k}(r)[\rho^D_{T,\ell_1}(\vec{r}_1)Y_k(\hat{r})]_{\ell_t m_{\ell_t}}
\end{aligned}
$$

The radial direct form factor from Section 2.1 gives

$$
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) = i^{-\pi}(-)^{\ell_1}\hat{\ell}_t^{-1} \int r^2 dr V^D_{t_1 s_1 k}(r) \int r_1^2 dr_1 \rho^D_{P,k\ell_t\ell_1}(r_a, r_1, r)\rho^D_{T,\ell_1}(r_1)
$$

See Fig. 9 for the breakup of $\vec{r}'_2$ into $\vec{r}_1$ and $\vec{r}_a$. The integration over dr_1 becomes

$$
\begin{aligned}
Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) &= \int r_1^2 dr_1 \rho^D_{P,k\ell_t\ell_1}(r_a, r_1, r)\rho^D_{T,\ell_1}(r_1) \\
&= \int r_1^2 dr_1 d\mu \rho^D_{T,\ell_1}(r_1)\frac{2\pi}{\hat{k}^2}\sum_m \hat{\ell}_t(\ell_t 0 \ell_1 m|km)\rho^D_{P,k\ell_t\ell_1}(r'_2, r)Y_{km}(\theta'_2, 0)Y^*_{\ell_1 m}(\theta, 0) \\
(r'_2)^2 &= r_1^2 + r_a^2 - 2r_1 r_a \mu, \\
\mu \equiv \cos\theta &= \hat{r}_a \cdot \hat{r}_1 = \frac{r_a^2 + r_1^2 - (r'_2)^2}{2r_1 r_a}, \; d\mu = -\frac{r'_2 dr'_2}{r_1 r_a} \; (d\mu \to dr'_2) \\
\mu' \equiv \cos\theta'_2 &= \hat{r}_a \cdot \hat{r}'_2 = \frac{r_a - r_1\mu}{r'_2} = \frac{r_a - r_1\mu}{r} = \frac{r_a^2 + r^2 - r_1^2}{2rr_a}, \; d\mu' = -\frac{r_1 dr_1}{rr_a} \; (dr_1 \to d\mu')
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) &= \int r_1^2 dr_1 \int (-\frac{r'_2 dr'_2}{r_1 r_a})\rho^D_{T,\ell_1}(r_1)\frac{2\pi}{\hat{k}^2} \\
&\times \sum_m \hat{\ell}_t(\ell_t 0 \ell_1 m|km)\hat{k}(-)^k \frac{\delta(r'_2 - r)}{r^2} Y_{km}(\theta'_2, 0)Y^*_{\ell_1 m}(\theta, 0) \\
&= \int r_1^2 dr_1 \rho^D_{T,\ell_1}(r_1)\frac{2\pi}{\hat{k}^2} \\
&\times \sum_m \hat{\ell}_t(\ell_t 0 \ell_1 m|km)\hat{k}(-)^k(\frac{-1}{r_1 r_a r})Y_{km}(\theta'_2, 0)Y^*_{\ell_1 m}(\theta, 0) \\
&= \int (-r_1 r_a r) d\mu' \rho^D_{T,\ell_1}(r_1)\frac{2\pi}{\hat{k}^2} \\
&\times \sum_m \hat{\ell}_t(\ell_t 0 \ell_1 m|km)\hat{k}(-)^k(\frac{-1}{r_1 r_a r})Y_{km}(\theta'_2, 0)Y^*_{\ell_1 m}(\theta, 0) \\
&= \frac{2\pi}{\hat{\ell}_1}\sum_m \hat{\ell}_t(-)^{\ell_t}(\ell_t 0 k m|\ell_1 m)(-)^k \int d\mu' \rho^D_{T,\ell_1}(r_1)Y_{km}(\theta'_2, 0)Y^*_{\ell_1 m}(\theta, 0)
\end{aligned}
$$

We finally obtain the direct form factor,

$$
\begin{aligned}
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi}(-)^{\ell_1+k-\ell_t} 2\pi \hat{\ell}_1^{-1} \sum_m (\ell_t 0 k m|\ell_1 m) \\
&\times \int r^2 dr V^D_{t_1 s_1 k}(r) \int d\mu' \rho^D_{T,\ell_1}(r_1) Y_{km}(\theta_2', 0) Y^*_{\ell_1 m}(\theta, 0)
\end{aligned}
$$

comparing that for the composite particle,

$$
\begin{aligned}
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi}(-)^{\ell_1} \hat{\ell}_t^{-1} \int r^2 dr V^D_{t_1 s_1 k}(r) \int r_1^2 dr_1 \rho^D_{P,k\ell_t\ell_1}(r_a, r_1, r) \rho^D_{T,\ell_1}(r_1) \\
\rho^D_{P,k\ell_t\ell_1} &= \frac{2\pi}{\hat{k}^2} \sum_m \hat{\ell}_t (\ell_t 0 \ell_1 m|km) \int \rho^D_{P,k\ell_t\ell_1}(r_a, r_1, \mu, r) Y_{km}(\theta_2', 0) Y^*_{\ell_1 m}(\theta, 0) d\mu
\end{aligned}
$$

Another way to calculate is that

$$
\begin{aligned}
\int d\vec{r}_1 \rho^D_T(\vec{r}_1) \rho^E_P(\vec{r}_2) &\rightarrow \int d\vec{r}_1 \rho^D_T(\vec{r}_1) \delta(\vec{r}_2) \\
&= \int d\vec{r}_1 \rho^D_T(\vec{r}_1) \delta(\vec{r}_1 - \vec{r}_a + \vec{r}) \\
&= \rho^D_T(\vec{r}_a - \vec{r})
\end{aligned}
$$

where we use $\vec{r}_2' = \vec{r}_1 - \vec{r}_b + \vec{r}$. Therefore $Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r)$ is nothing but the multipole expansion coefficients of $\rho^E_T(\vec{r}_a - \vec{r})$

$$
\begin{aligned}
\rho^D_{T,\ell_1 m_{\ell_1}}(\vec{r}_a - \vec{r}) &= \sum_{\ell_a \lambda} \rho^D_{T,\ell_1 \ell_b \lambda}(r_a, r) [Y_{\ell_a}(\hat{r}_a) Y_\lambda(\hat{r})]_{\ell_1 m_{\ell_1}} \\
&= \sum_{\ell_a \lambda} Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) [Y_{\ell_t}(\hat{r}_a) Y_k(\hat{r})]_{\ell_1 m_{\ell_1}} \\
Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) &= \frac{2\pi}{\hat{\ell}_1} \sum_m \hat{\ell}_t (-)^{\ell_t} (\ell_t 0 k m|\ell_1 m)(-)^k \int d\mu' \rho^D_{T,\ell_1}(r_1) Y_{km}(\theta_2', 0) Y^*_{\ell_1 m}(\theta, 0)
\end{aligned}
$$

It agrees with the above limiting form factor.

We now summarize the radial direct form factors as

$$
\begin{aligned}
f^D_{t_1 s_1 \ell_1 k \ell_t}(r_a) &= i^{-\pi}(-)^{\ell_1} \hat{\ell}_t^{-1} \int r^2 dr V^D_{t_1 s_1 k}(r) Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) \\
Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r) &= \int r_1^2 dr_1 \rho^D_{T,\ell_1}(r_1) \frac{2\pi}{\hat{k}^2} \\
&\times \sum_m \hat{\ell}_t (\ell_t 0 \ell_1 m|km) \hat{k} (-)^k \left(\frac{-1}{r_1 r_a r}\right) Y_{km}(\theta_2', 0) Y^*_{\ell_1 m}(\theta, 0) \\
\mu \equiv \cos\theta &= \frac{r_a^2 + r_1^2 - r^2}{2 r_1 r_a}, \quad \mu' \equiv \cos\theta_2' = \frac{r_a^2 + r^2 - r_1^2}{2 r r_a}
\end{aligned}
$$

The above equation is what we calculated in the computer program, i.e., the second step for $Q^D_{t_1 s_1 \ell_1 k \ell_t}(r_a, r)$.

2) Exchange form factor

We adopt the following limits for the exchange form factor

$$\begin{aligned}\rho_P(\vec{r}_2,\vec{r}_2') &= \delta(\vec{r}_2)\\ \rho_{P,\lambda_2}(r_2',r) &= \hat{\lambda}_2(-)^{\lambda_2}\delta(r_2'-r)/r^2\end{aligned}$$

The exchange form factor of Section 1.9 gives [Eq.(38b)],

$$\begin{aligned}F^E_{t_1s_1\ell_1k\ell_tm_{\ell_t}}(\vec{r}_b,\vec{r}_a) &= J\int d\vec{r}_1\rho^E_P(\vec{r}_2,\vec{r}_2')V^E_{t_1s_1k}(r)[\rho^E_{T,\ell_1}(\vec{r}_1,\vec{r}_1')Y_k(\hat{r})]_{\ell_tm_{\ell_t}}\\ &= J\int d\vec{r}_1\delta(\vec{r}_2)V^E_{t_1s_1k}(r)[\rho^E_{T,\ell_1}(\vec{r}_1,\vec{r}_1')Y_k(\hat{r})]_{\ell_tm_{\ell_t}}\\ &= J\; V^E_{t_1s_1k}(r)[\rho^E_{T,\ell_1}(\vec{r}_b-\vec{r},\vec{r}_b)Y_k(\hat{r})]_{\ell_tm_{\ell_t}}\end{aligned}$$

Here remembering that

$$\vec{r}_2=\vec{r}_1-\vec{r}_b+\vec{r},\qquad \vec{r}_1'=\vec{r}_1+\vec{r}$$

(See Fig.2.), we use in this limit,

$$\vec{r}_1\to\vec{r}_b-\vec{r},\qquad \vec{r}_1'\to\vec{r}_b-\vec{r}+\vec{r}=\vec{r}_b.$$

We now calculate the projectile density for the radial exchange form factor from Section 2.2 gives

$$\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r) = \frac{2\pi}{\hat{\lambda}_2^2}\sum_m\hat{\ell}(\ell0\ell_cm|\lambda_2m)\int\rho^E_{P,\lambda_2\ell\ell_q}(r_b,r_1,\mu,r)Y_{\lambda_2m}(\theta_2',0)Y^*_{\ell_cm}(\theta,0)d\mu$$

where the separation procedure can be seen in Fig.9,

$$\begin{aligned}(r_2')^2 &= r_1^2+r_b^2-2r_1r_b\mu,\\ \mu\equiv\cos\theta &= \hat{r}_b\cdot\hat{r}_1=\frac{r_b^2+r_1^2-(r_2')^2}{2r_1r_b},\quad d\mu=-\frac{r_2'dr_2'}{r_1r_b}\\ \mu'\equiv\cos\theta' &= \hat{r}_b\cdot\hat{r}_2'=\frac{r_b-r_1\mu}{r_2'}=\frac{r_b-r_1\mu}{r}=\frac{r_b^2+r^2-r_1^2}{2rr_b},\quad d\mu'=-\frac{r_1dr_1}{rr_b}\\ \rho_{P,\lambda_2}(r_2',r) &= \hat{\lambda}_2(-)^{\lambda_2}\delta(r_2'-r)/r^2\end{aligned}$$

Putting them together and performing the integration over $d\mu$ give

$$\begin{aligned}\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r) &= \frac{2\pi}{\hat{\lambda}_2^2}\sum_m\hat{\ell}(\ell0\ell_cm|\lambda_2m)\hat{\lambda}_2(-)^{\lambda_2}(-\frac{1}{r_1r_br})Y_{\lambda_2m}(\mu')Y^*_{\ell_cm}(\mu)\\ \mu' &= \frac{r_b-r_1\mu}{r}=\frac{r_b^2+r^2-r_1^2}{2rr_b},\\ \mu &= \frac{r_b^2+r_1^2-r^2}{2r_1r_b}\end{aligned}$$

Another way to calculate is that

$$\begin{aligned}\int d\vec{r}_1\rho^E_T(\vec{r}_1,\vec{r}_1')\rho^E_P(\vec{r}_2,\vec{r}_2') &\to \int d\vec{r}_1\rho^E_T(\vec{r}_1,\vec{r}_1')\delta(\vec{r}_2')\\ &= \int d\vec{r}_1\rho^E_T(\vec{r}_1,\vec{r}_1')\delta(\vec{r}_1-\vec{r}_b+\vec{r})\\ &= \rho^E_T(\vec{r}_b-\vec{r},\vec{r}_1+\vec{r})\\ &= \rho^E_T(\vec{r}_b-\vec{r},\vec{r}_b)\end{aligned}$$

where we use $\vec{r}_2' = \vec{r}_1 - \vec{r}_b + \vec{r}$, and $\vec{r}_1' = \vec{r}_1 + \vec{r} = \vec{r}_b$. Our goal is to obtain G-factor in (r_b, r) like

$$G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) \quad = \quad r^{-k} V^E_{t_1 s_1 k}(r) \rho^E_{T,\ell_1 \ell \lambda}(r_b, r)$$

where $\rho^E_T(r_b, r)$ is nothing but the multipole expansion coefficients of $\rho^E_T(\vec{r}_b - \vec{r}, \vec{r}_b)$, namely the particle state wave function $R_{\ell_p}(r_1)$ that depends on $(\vec{r}_1 = \vec{r}_b - \vec{r})$ must be expanded.

$$\begin{aligned}
\rho^E_{T,\ell_1 m_{\ell_1}}(\vec{r}_b - \vec{r}, \vec{r}_b) &= \sum_{\ell\lambda} \rho^E_{T,\ell_1\ell\lambda}(r_b, r)[Y_\ell(\hat{r}_b) Y_\lambda(\hat{r})]_{\ell_1 m_{\ell_1}} \\
&= \sum_{ph} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B || [\hat{a}^\dagger_{j_p \nu_p} \hat{a}_{\tilde{j_h \nu_h}}]_{j_t} || I_A > [\phi_{\ell_p}(\vec{r}_1) \phi^*_{\tilde{\ell}_h}(\vec{r}_b)]_{\ell_1 m_{\ell_1}} \\
&= \sum_{ph} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B || [\hat{a}^\dagger_{j_p \nu_p} \hat{a}_{\tilde{j_h \nu_h}}]_{j_t} || I_A > \\
&\qquad i^{\ell_p + \ell_h - \pi} R_{\ell_p}(r_1) R_{\ell_h}(r_b) (\ell_p m_p \ell_h, -m_h | \ell_1 m_{\ell_1}) Y_{\ell_p m_p}(\hat{r}_1) Y^*_{\ell_h, -m_h}(\hat{r}_b)
\end{aligned}$$

We now have

$$\begin{aligned}
R_{\ell_p}(r_1) Y_{\ell_p, m_p}(\hat{r}_1) &= \sqrt{4\pi} \sum_{\eta\lambda} R_{\ell_p \eta \lambda}(r_b, r) \sum_{\nu\mu} (\eta\nu\lambda\mu | \ell_p m_p) Y_{\eta\nu}(\hat{r}_b) Y_{\lambda\mu}(\hat{r}) \\
R_{\ell_p \eta\lambda}(r_b, r) &= \frac{2\pi}{2\ell_p + 1} \sum_m \hat{\eta} (\eta 0 \lambda m | \ell_p m) \int R_{\ell_p}(r_1) Y_{\ell_p m}(\theta, 0) Y^*_{\lambda m}(\theta', 0) d\mu' \\
&\qquad (\mu = \cos\theta = \hat{r}_1 \cdot \hat{r}_b, \quad \mu' = \cos\theta' = \hat{r}_b \cdot \hat{r}) \\
&\qquad (r_1^2 = r_b^2 + r^2 - 2 r_b r \mu', \quad \mu = (r_b - r\mu')/r_1) \\
Y^*_{\ell_h m_h}(\hat{r}_b) Y_{\eta\nu}(\hat{r}_b) &= \sum_\ell \frac{\hat{\ell}_h \hat{\eta}}{\sqrt{4\pi} \hat{\ell}} (\ell_h 0 \eta 0 | \ell 0)(\ell_h m_h \eta\nu | \ell m) Y_{\ell m}(\hat{r}_b) \\
Y_{\ell m}(\hat{r}_b) Y_{\lambda\mu}(\hat{r}) &= \sum_{\ell_1 m_1} (\ell m \lambda \mu | \ell_1 m_1) [Y_\ell(\hat{r}_b) Y_\lambda(\hat{r})]_{\ell_1 m_1}
\end{aligned}$$

Combining CG's gives

$$\begin{aligned}
\text{Geometry} &= \sqrt{4\pi} \frac{\hat{\ell}_h \hat{\eta}}{\sqrt{4\pi}\hat{\ell}} (\ell_h 0 \eta 0 | \ell 0) \\
&\quad \sum_{all\ m} (\ell_p m_p \ell_h m_h | \ell_1 m_{\ell_1})(\eta\nu\lambda\mu | \ell_p m_p)(\ell_h m_h \eta\nu | \ell m)(\ell m \lambda\mu | \ell_1 m_1) \\
&= (-)^{\ell_p + \ell_h - \ell_1} (-)^\eta \hat{\ell}_p \hat{\ell} \hat{\eta} (\eta 0 \ell 0 | \ell_h 0) \ W(\eta \ell \ell_1 \ell_h; \ell_p \lambda)
\end{aligned}$$

Finally we obtain the non-local target density,

$$\begin{aligned}
\rho^E_{T,\ell_1 m_{\ell_1}}(\vec{r}_b - \vec{r}, \vec{r}_b) &= \sum_{\ell\lambda} \rho^E_{T,\ell_1\ell\lambda}(r_b, r)[Y_\ell(\hat{r}_b) Y_\lambda(\hat{r})]_{\ell_1 m_{\ell_1}} \\
\rho^E_{T,\ell_1\ell\lambda}(r_b, r) &= \sum_{ph,\eta} i^{\ell_p + \ell_h - \pi} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B || [\hat{a}^\dagger_{j_p \nu_p} \hat{a}_{\tilde{j_h \nu_h}}]_{j_t} || I_A > R_{\ell_h}(r_b) \\
&\qquad \times \ (-)^{\ell_p + \ell_h - \ell_1} (-)^\eta \hat{\ell}_p \hat{\ell} \hat{\eta} (\eta 0 \ell 0 | \ell_h 0) \ W(\eta \ell \ell_1 \ell_h; \ell_p \lambda) \\
&\qquad \times \ \frac{2\pi}{2\ell_p + 1} \sum_m \hat{\eta} (\eta 0 \lambda m | \ell_p m) \int R_{\ell_p}(r_1) Y_{\ell_p m}(\theta, 0) Y^*_{\lambda m}(\theta', 0) d\mu'
\end{aligned}$$

We now summarize the radial exchange form factors for the nucleon-nucleus scattering as

$$
\begin{aligned}
f^E_{t_1s_1\ell_1k\ell_t,\ell_b\ell_a}(r_b,r_a) &= J\ 4\pi\ m_a^k \sum_{\lambda_a\lambda_b\ell_\alpha\ell_\beta}[\frac{(2k+1)!}{(2\lambda_a+1)!(2\lambda_b+1)!}]^{1/2}\delta_{\lambda_a+\lambda_b,k}\ (-r_a)^{\lambda_a}(r_b)^{\lambda_b} \\
&\times\ X(\ell_\alpha\lambda_a\ell_a,\ell_\beta\lambda_b\ell_b;\ell_1k\ell_t)d_{\ell_\alpha\lambda_a\ell_a}d_{\ell_\beta\lambda_b\ell_b}\ c_{t_1s_1\ell_1k,\ell_\alpha\ell_\beta}(r_b,r_a) \\
c_{t_1s_1\ell_1k,\ell_\alpha\ell_\beta}(r_b,r_a) &= \frac{2\pi}{\hat{\ell}_1^2}\sum_{m_{\ell_1}}\hat{\ell}_\beta(\ell_\alpha m_{\ell_1}\ell_\beta 0|\ell_1m_{\ell_1})\sum_{\ell\lambda}\hat{\ell}(\ell 0\lambda m_{\ell_1}|\ell_1m_{\ell_1}) \\
&\times\ \int d\mu G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)Y_{\lambda m_{\ell_1}}(\theta',\pi)Y^*_{\ell_\alpha m_{\ell_1}}(\theta,0)
\end{aligned}
$$

where $G^k_{t_1s_1\ell_1\ell_b\lambda}(r_b,r)$ can be obtained in two ways, i.e.

$$
\begin{aligned}
G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r) &= r^{-k}V^E_{t_1s_1k}(r)\rho^E_{T,\ell_1\ell\lambda}(r_b,r) \\
\rho^E_{T,\ell_1\ell\lambda}(r_b,r) &= \sum_{ph,\eta} i^{\ell_p+\ell_h-\pi}X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1s_1j_t)\ <I_B||[\hat{a}^\dagger_{j_p\nu_p}\hat{a}_{\tilde{j_h\nu_h}}]_{j_t}||I_A> R_{\ell_h}(r_b) \\
&\times\ (-)^{\ell_p+\ell_h-\ell_1}(-)^\eta\hat{\ell}_p\hat{\ell}\hat{\eta}(\eta 0\ell 0|\ell_h 0)\ W(\eta\ell\ell_1\ell_h;\ell_p\lambda) \\
&\times\ \frac{2\pi}{2\ell_p+1}\sum_m\hat{\eta}(\eta 0\lambda m|\ell_p m)\int R_{\ell_p}(r_1)Y_{\ell_p m}(\theta,0)Y^*_{\lambda m}(\theta',0)d\mu'
\end{aligned}
$$

or

$$
\begin{aligned}
G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r) &= \frac{1}{\sqrt{4\pi}}r^{-k}V^E_{t_1s_1k}(r)\sum_{\lambda_1\lambda_2\ell_c}(-)^\ell\hat{\lambda}_1\hat{\lambda}_2(\lambda_1 0\lambda_2 0|\lambda 0)W(\lambda_1\lambda_2\ell_1\ell;\lambda\ell_c) \\
&\times\ \int r_1^2dr_1\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r)\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r) \\
\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,r) &= \frac{2\pi}{\hat{\lambda}_2^2}\sum_m\hat{\ell}(\ell 0\ell_c m|\lambda_2 m)\int\rho^E_{P,\lambda_2\ell\ell_c}(r_b,r_1,\mu,r)Y_{\lambda_2 m}(\theta_2',0)Y^*_{\ell_c m}(\theta,0)d\mu \\
\mu &= \frac{r_b^2+r_1^2-r^2}{2r_1r_b},\quad \mu' = \frac{r_b-r_1\mu}{r} \\
\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1,r) &= \sum_{ph,\eta_1} i^{\ell_p+\ell_h-\pi}X(\ell_p\frac{1}{2}j_p,\ell_h\frac{1}{2}j_h;\ell_1s_1j_t)\ <I_B||[\hat{a}^\dagger_{j_p\nu_p}\hat{a}_{j_h\nu_h}]_{j_t}||I_A> R_{\ell_p}(r_1) \\
&\times\ (-)^{\eta_1}\hat{\ell}_h\hat{\ell}_c\hat{\eta}_1(\ell_c 0\eta_1 0|\ell_p 0)\ W(\ell_c\eta_1\ell_1\ell_h;\ell_p\lambda_1) \\
&\times\ \frac{2\pi}{\hat{\ell}_h^2}\sum_{m_1}\hat{\eta}_1(\eta_1 0\lambda_1 m_1|\ell_h m_1)\int R_{\ell_h}(r_1')Y_{\ell_h m_1}(\theta,0)Y^*_{\lambda_1 m_1}(\theta',0)d\mu'
\end{aligned}
$$

In the computer program, we choose the latter one.

3.2 Exchange form factor in the no-recoil approximation

The no-recoil approximation was originally invented to simplify the cross section calculations for heavy-ion induced one and two nucleon transfer reactions.[5] The essence of the approximation is to ignore the recoil momentum the target receives in the transfer process. In the same spirit we neglect here the recoil momenta which projectile and target pick up in the knock-on exchange process.

Formally this approximation is obtained by replacing $\chi_a^{(+)}(\vec{k}_a, \vec{r}_a)$ in [Eq.(15)] through $\chi_a^{(+)}(\vec{k}_a, \vec{r}_b)$, i.e., ignoring the difference between the vectors $\vec{r}_a$ and $\vec{r}_b$. The exchange transition amplitude, [Eq.(14)], becomes [Eq.(40)],

$$\begin{aligned} T^{E,NR}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}} &= \int d\vec{r}_b \chi_b^{(-)*}(\vec{k}_b, \vec{r}_b) F^{E,NR}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b) \chi_a^{(+)}(\vec{k}_a, \vec{r}_b) \\ F^{E,NR}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b) &= J^{-1} \int d\vec{r}\, F^{E}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b, \vec{r}) \\ &= \int d\vec{r} \int d\vec{r}_1 \rho^E_P(\vec{r}_2, \vec{r}_2^{\,\prime}) V^E_{t_1 s_1 k}(r) [\rho^E_{T,\ell_1}(\vec{r}_1, \vec{r}_1^{\,\prime}) Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \end{aligned}$$

This has a similar structure as [Eq.(14)] for the direct amplitude,

$$F^{D}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_a) = \int d\vec{r}_1 \int d\vec{r}_2 \rho^D_P(\vec{r}_2) V^D_{t_1 s_1 k}(r) [\rho^D_{T,\ell_1}(\vec{r}_1) Y_k(\hat{r})]_{\ell_t m_{\ell_t}}$$

We now try to obtain the no-recoil radial exchange form factor defined as usual, (We ignore the superscript "E".)

$$F^{NR}_{t_1 s_1 \ell_1 k \ell_t m_{\ell_t}}(\vec{r}_b) = J^{-1} f^{NR}_{t_1 s_1 \ell_1 k \ell_t}(r_b) Y_{\ell_t m_{\ell_t}}(\hat{r}_b) i^{\pi} = \int d\vec{r}\, f^{E}_{t_1 s_1 \ell_1 k \ell_t}(r_b)$$

Remembering that

$$\begin{aligned} f^E_{t_1 s_1 \ell_1 k \ell_t}(r_b, r_a) &= J i^{-\pi} \int d\hat{r}_a \int d\hat{r}_b \int d\vec{r}_1 \, [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]^*_{\ell_t m_{\ell_t}} \\ &\quad \times \rho^E_P(\vec{r}_2, \vec{r}_2^{\,\prime}) V^E_{t_1 s_1 k}(r) [\rho^E_{T,\ell_1}(\vec{r}_1, \vec{r}_1^{\,\prime}) Y_k(\hat{r})]_{\ell_t m_{\ell_t}} \\ c_{t_1 s_1 \ell_1 m_{\ell_1}}(\vec{r}_b, \vec{r}) &\equiv \frac{1}{\sqrt{4\pi}} r^{-k} V^E_{t_1 s_1 k}(r) \int d\vec{r}_1 \rho^E_P(\vec{r}_2, \vec{r}_2^{\,\prime}) \rho^E_{T,\ell_1 m_{\ell_1}}(\vec{r}_1, \vec{r}_1^{\,\prime}) \\ &= \sum_{\ell_b \lambda} i^{\pi} G^k_{t_1 s_1 \ell_1 \ell_b \lambda}(r_b, r) [Y_\lambda(\hat{r}) Y_\ell(\hat{r}_b)]_{\ell_1 m_{\ell_1}} \\ G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) &= \frac{1}{\sqrt{4\pi}} r^{-k} V^E_{t_1 s_1 k}(r) \sum_{\lambda_1 \lambda_2 \ell_c} (-)^{\ell} \hat{\lambda}_1 \hat{\lambda}_2 (\lambda_1 0 \lambda_2 0 | \lambda 0) W(\lambda_1 \lambda_2 \ell_1 \ell; \lambda \ell_c) \\ &\quad \times \int r_1^2 dr_1 \rho^E_{P,\lambda_2 \ell \ell_c}(r_b, r_1, r) \rho^E_{T,\ell_1 \lambda_1 \ell_c}(r_1, r) \end{aligned}$$

we now calculate

$$\begin{aligned} F^{NR}_{t_1 s_1 \ell_1 k \ell_t}(\vec{r}_b) &= \sqrt{4\pi} \int d\vec{r} \sum_{q, m_{\ell_1}} (k q \ell_1 m_{\ell_1} | \ell_t m_{\ell_t}) r^k Y_{kq}(\hat{r}) \sum_{\ell_b \lambda} G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) [Y_\lambda(\hat{r}) Y_\ell(\hat{r}_b)]_{\ell_1 m_{\ell_1}} \\ &= \sqrt{4\pi} \sum_{\ell_b \lambda} \int d\hat{r} \sum_{q, m_{\ell_1}} (k q \ell_1 m_{\ell_1} | \ell_t m_{\ell_t}) Y_{kq}(\hat{r}) Y_{\lambda\mu}(\hat{r}) (\lambda \mu \ell m_\ell | \ell_1 m_{\ell_1}) Y_{\ell_b m_{\ell_b}}(\hat{r}_b) \\ &\quad \int dr\, r^{k+2} G^k_{t_1 s_1 \ell_1 \ell \lambda}(r_b, r) \\ &= \sqrt{4\pi} (-)^k \hat{\ell}_1 \hat{\ell}_t^{-1} Y_{\ell_t m_{\ell_t}}(\hat{r}_b) \int dr\, r^{k+2} G^k_{t_1 s_1 \ell_1 \ell_t \lambda}(r_b, r) \\ f^{NR}_{t_1 s_1 \ell_1 k \ell_t}(r_b) &= \sqrt{4\pi} (-)^k \hat{\ell}_1 \hat{\ell}_t^{-1} \int dr\, r^{k+2} G^k_{t_1 s_1 \ell_1 \ell_t \lambda}(r_b, r) \end{aligned}$$

[5] G. R. Satchler, "Direct Nuclear Reactions" (1983), Sec. 6.14, 15.4.3, and 16.5.3.

where we use

$$\int d\hat{r} Y_{kq}(\hat{r}) Y_{\lambda\mu}(\hat{r}) = (-)^q \delta_{k,\lambda}\delta_{q,-\mu}$$

$$\sum_{q,m_{\ell_1}} (-)^q (kq\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})(\lambda\mu\ell m_\ell|\ell_1 m_{\ell_1}) \quad = \quad (-)^k \hat{\ell}_1 \hat{\ell}^{-1} \delta_{\ell_t,\ell}$$

We now summarize the radial exchange form factors in the no-recoil approximation as

$$\begin{aligned}
f^{NR}_{t_1 s_1 \ell_1 k \ell_t}(r_b) &= \sqrt{4\pi}(-)^k \hat{\ell}_1 \hat{\ell}_t^{-1} \int dr\ r^{k+2} G^k_{t_1 s_1 \ell_1 \ell_t k}(r_b, r)\delta_{k,\lambda}\delta_{\ell_t,\ell} \\
G^k_{t_1 s_1 \ell_1 \ell_t k}(r_b, r) &= \frac{1}{\sqrt{4\pi}} r^{-k} V^E_{t_1 s_1 k}(r) \sum_{\lambda_1 \lambda_2 \ell_c} (-)^{\ell_t} \hat{\lambda}_1 \hat{\lambda}_2 (\lambda_1 0 \lambda_2 0|\lambda 0) W(\lambda_1 \lambda_2 \ell_1 \ell_t; \lambda \ell_c) \\
&\qquad \times \int r_1^2 dr_1 \rho^E_{P,\lambda_2 \ell_t \ell_c}(r_b, r_1, r) \rho^E_{T,\ell_1 \lambda_1 \ell_c}(r_1, r) \\
\rho^E_{P,\lambda_2 \ell_t \ell_c}(r_b, r_1, r) &= \frac{2\pi}{\hat{\lambda}_2^2} \sum_m \hat{\ell}_t (\ell_t 0 \ell_c m|\lambda_2 m) \int \rho^E_{P,\lambda_2 \ell_t \ell_c}(r_b, r_1, \mu, r) Y_{\lambda_2 m}(\theta'_2, 0) Y^*_{\ell_c m}(\theta, 0) d\mu \\
\rho^E_{T,\ell_1 \lambda_1 \ell_c}(r_1, r) &= \sum_{ph,\eta_1} i^{\ell_p + \ell_h - \pi} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t) \ < I_B || [\hat{a}^\dagger_{j_p \nu_p} \hat{a}_{j_h \nu_h}]_{j_t} || I_A > R_{\ell_p}(r_1) \\
&\quad \times\ (-)^{\eta_1} \hat{\ell}_h \hat{\ell}_c \hat{\eta}_1 (\ell_c 0 \eta_1 0|\ell_p 0)\ W(\ell_c \eta_1 \ell_1 \ell_h; \ell_p \lambda_1) \\
&\quad \times\ \frac{2\pi}{\hat{\ell}_h^2} \sum_{m_1} \hat{\eta}_1 (\eta_1 0 \lambda_1 m_1|\ell_h m_1) \int R_{\ell_h}(r'_1) Y_{\ell_h m_1}(\theta, 0) Y^*_{\lambda_1 m_1}(\theta', 0) d\mu'
\end{aligned}$$

3.3 Exchange form factor in the plane wave approximation

In the plane wave approximation, the recoil effect[6] is described by a recoil factor $\exp(-i\alpha\vec{k}_a\cdot r/a)$. A simple reason is that in the plane wave approximation the incoming and outgoing waves are described by

$$\exp(i\vec{k}_a\cdot\vec{r}_a - i\vec{k}_b\cdot\vec{r}_b) \;=\; \exp[i(\vec{k}_a-\vec{k}_b)\cdot\vec{r}_b]\exp[-i\vec{k}_a\cdot\vec{r}/a]$$

where we use $\vec{r}_a=\vec{r}_b-\vec{r}/a$. Obviously $-\vec{k}_a/a$ is the change of linear momentum between the exchanged particles.

A possible improvement of the no-recoil approximation is then to replace f^{NR} by the following f^{PW} that takes into account the recoil factor within the plane wave approximation,

$$F^{PW}_{t_1s_1\ell_1k\ell_tm_{\ell_t}}(\vec{r}_b) \;=\; J^{-1}f^{PW}_{t_1s_1\ell_1k\ell_t}(r_b)Y_{\ell_tm_{\ell_t}}(\hat{r}_b)i^{\pi} = \int d\vec{r}\; f^{E}_{t_1s_1\ell_1k\ell_t}(r_b)\exp(-i\alpha\vec{k}_a\cdot\vec{r}/a)$$

where a parameter α in the recoil factor is treated as an adjustable parameter. We fit it such that the resultant approximate cross section reproduces the exact cross section $\sigma(E)$ as closely as possible. It has turned out that a close fit is obtained with $\alpha=1$.[7]

We first introduce the partial expansion of the recoil factor

$$\begin{aligned}\exp(-i\alpha\vec{k}_a\cdot\vec{r}/a) &= 4\pi\sum_{\ell_rm_r}(-i)^{\ell_r}j_{\ell_r}(\alpha k_ar)Y^*_{\ell_rm_r}(\hat{r})Y_{\ell_rm_r}(\hat{k}_a)\\ &= \sqrt{4\pi}\sum_{\ell_r}\hat{\ell}_r(-i)^{\ell_r}j_{\ell_r}(\alpha k_ar/a)Y^*_{\ell_r0}(\hat{r})\end{aligned}$$

where we set $Y_{\ell_rm_r}(\hat{k}_a)=\hat{\ell}_r/\sqrt{4\pi}\;\delta_{m_r,0}$. We now calculate

$$\begin{aligned}F^{PW}_{t_1s_1\ell_1k\ell_t}(\vec{r}_b) &= \int d\vec{r}\; f^{E}_{t_1s_1\ell_1k\ell_t}(r_b)\exp(-i\alpha\vec{k}_a\cdot\vec{r}/a)\\
&= 4\pi\int d\vec{r}\sum_{q,m_{\ell_1}}(kq\ell_1m_{\ell_1}|\ell_tm_{\ell_t})r^kY_{kq}(\hat{r})\\
&\qquad\times\sum_{\ell_b\lambda\ell_r}G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)[Y_\lambda(\hat{r})Y_\ell(\hat{r}_b)]_{\ell_1m_{\ell_1}}\hat{\ell}_r(-i)^{\ell_r}j_{\ell_r}(\alpha k_ar/a)Y^*_{\ell_r0}(\hat{r})\\
&= 4\pi\sum_{\ell_b\lambda\ell_r}\hat{\ell}_r(-i)^{\ell_r}\int dr\; r^{k+2}G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)j_{\ell_r}(\alpha k_ar/a)\\
&\qquad\sum_{q,m_{\ell_1}}Y_{\ell_bm_{\ell_b}}(\hat{r}_b)(kq\ell_1m_{\ell_1}|\ell_tm_{\ell_t})(\lambda\mu\ell m_\ell|\ell_1m_{\ell_1})\int d\hat{r}Y_{kq}(\hat{r})Y_{\lambda\mu}(\hat{r})Y^*_{\ell_r0}(\hat{r})\\
&= \sqrt{4\pi}\sum_{\ell\lambda\ell_r}\hat{k}\hat{\lambda}(k0\lambda0|\ell_r0)(-i)^{\ell_r}\int dr\; r^{k+2}G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)j_{\ell_r}(\alpha k_ar/a)\\
&\qquad\sum_{q,m_{\ell_1}}Y_{\ell m_\ell}(\hat{r}_b)(kq\ell_1m_{\ell_1}|\ell_tm_{\ell_t})(\lambda\mu\ell m_\ell|\ell_1m_{\ell_1})(kq\lambda\mu|\ell_r0)\\
&= \sqrt{4\pi}\sum_{\ell\lambda\ell_r}(-i)^{\ell_r}\hat{k}\hat{\lambda}(k0\lambda0|\ell_r0)\hat{\ell}_1\hat{\ell}_rW(\ell\lambda\ell_tk:\ell_1\ell_r)(-)^{k+\ell_1-\ell_t}(\ell m_\ell\ell_r0|\ell_tm_{\ell_t})\\
&\qquad Y_{\ell m_\ell}(\hat{r}_b)\int dr\; r^{k+2}G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)j_{\ell_r}(\alpha k_ar/a)\\
&= \sqrt{4\pi}\sum_{\ell\lambda\ell_r}i^{\pi-\ell_r-\ell}\hat{k}\hat{\lambda}(k0\lambda0|\ell_r0)\hat{\ell}_1\hat{\ell}_rW(\ell\lambda\ell_tk:\ell_1\ell_r)(-)^{k+\ell_1-\ell_t}(\ell m_{\ell_t}\ell_r0|\ell_tm_{\ell_t})\\
&\qquad i^\ell Y_{\ell m_{\ell_t}}(\hat{r}_b)\int dr\; r^{k+2}G^k_{t_1s_1\ell_1\ell\lambda}(r_b,r)j_{\ell_r}(\alpha k_ar/a)\end{aligned}$$

[6] T.Tamura, Phys. Rep. 14C, 59 (1974), Section 4.4.

[7] B. T. Kim, D. P. Knobles, S. A. Stotts, and T. Udagawa, Phys. Rev. C61, 044611 (2000).

where we use

$$
\begin{aligned}
\int d\hat{r} Y_{kq}(\hat{r}) Y_{\lambda\mu}(\hat{r}) Y^*_{\ell_r 0}(\hat{r}) &= \frac{1}{\sqrt{4\pi}} \hat{k}\hat{\lambda}\hat{\ell}_r^{-1} (k0\lambda 0|\ell_r 0)(kq\lambda\mu|\ell_r 0) \\
\sum_{q,m_{\ell_1}} (kq\ell_1 m_{\ell_1}|\ell_t m_{\ell_t})(\lambda\mu\ell m_\ell|\ell_1 m_{\ell_1})(kq\lambda\mu|\ell_r 0) &= \hat{\ell}_1\hat{\ell}_r W(\ell\lambda\ell_t k : \ell_1\ell_r)(-)^{k+\ell_1-\ell_t}(\ell m_\ell \ell_r 0|\ell_t m_{\ell_t})
\end{aligned}
$$

We see that this form factor goes back to no-recoil form factor by setting $\ell_r = 0$. Reminding that $j_0(x) \to 1$ as $x \to 0$, we have

$$
\begin{aligned}
F^{PW}_{t_1 s_1 \ell_1 k \ell_t}(\vec{r}_b) &\to \sqrt{4\pi} \sum_{\ell\lambda} i^{\pi-\ell} \hat{k}\hat{\lambda}(k0\lambda 0|00)\hat{\ell}_1 W(\ell\lambda\ell_t k : \ell_1 0)(-)^{k+\ell_1-\ell_t}(\ell_b m_{\ell_t} 00|\ell_t m_{\ell_t}) \\
& \quad i^{\ell_b} Y_{\ell m_\ell}(\hat{r}_b) \int dr\ r^{k+2} G^k_{t_1 s_1 \ell_1 \ell\lambda}(r_b, r) j_0(\alpha k_a r/a) \\
&= \sqrt{4\pi} \sum_{\ell\lambda} i^{\pi} \hat{k}^2 (-)^k \hat{k}^{-1} \hat{\ell}_1 (-)^{k+\ell_t-\ell_1} \hat{\ell}^{-1} \hat{\lambda}^{-1} \delta_{\lambda k}\delta_{\ell\ell_t}(-)^{k+\ell_1-\ell_t} Y_{\ell m_{\ell_t}}(\hat{r}_b) \\
& \quad \int dr\ r^{k+2} G^k_{t_1 s_1 \ell_1 \ell\lambda}(r_b, r) \\
&= \sqrt{4\pi}(-)^k \hat{\ell}_1 \hat{\ell}_t^{-1} Y_{\ell_t m_{\ell_t}}(\hat{r}_b) \int dr\ r^{k+2} G^k_{t_1 s_1 \ell_1 \ell_t k}(r_b, r) \\
&= F^{NR}_{t_1 s_1 \ell_1 k \ell_t}(\vec{r}_b)
\end{aligned}
$$

We now summarize the radial exchange form factors in the plane wave approximation as

$$
\begin{aligned}
f^{PW}_{t_1 s_1 \ell_1 k \ell_t}(r_b) &= \sqrt{4\pi} \sum_{\ell\lambda\ell_r} i^{\pi-\ell_r-\ell} \hat{k}\hat{\lambda}(k0\lambda 0|\ell_r 0)\hat{\ell}_1\hat{\ell}_r W(\ell\lambda\ell_t k : \ell_1\ell_r) \\
& \quad \times (-)^{k+\ell_1-\ell_t}(\ell m_{\ell_t}\ell_r 0|\ell_t m_{\ell_t}) \int dr\ r^{k+2} G^k_{t_1 s_1 \ell_1 \ell\lambda}(r_b, r) j_{\ell_r}(\alpha k_a r/a) \\
G^k_{t_1 s_1 \ell_1 \ell\lambda}(r_b, r) &= \frac{1}{\sqrt{4\pi}} r^{-k} V^E_{t_1 s_1 k}(r) \sum_{\lambda_1\lambda_2\ell_c} (-)^\ell \hat{\lambda}_1\hat{\lambda}_2(\lambda_1 0\lambda_2 0|\lambda 0) W(\lambda_1\lambda_2\ell_1\ell; \lambda\ell_c) \\
& \quad \times \int r_1^2 dr_1 \rho^E_{P,\lambda_2\ell\ell_c}(r_b, r_1, r)\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1, r) \\
\rho^E_{P,\lambda_2\ell\ell_c}(r_b, r_1, r) &= \frac{2\pi}{\hat{\lambda}_2^2} \sum_m \hat{\ell}(\ell 0\ell_c m|\lambda_2 m) \int \rho^E_{P,\lambda_2\ell\ell_c}(r_b, r_1, \mu, r) Y_{\lambda_2 m}(\theta'_2, 0) Y^*_{\ell_c m}(\theta, 0) d\mu \\
\rho^E_{T,\ell_1\lambda_1\ell_c}(r_1, r) &= \sum_{ph,\eta_1} i^{\ell_p+\ell_h-\pi} X(\ell_p \frac{1}{2} j_p, \ell_h \frac{1}{2} j_h; \ell_1 s_1 j_t)\ \ < I_B || [\hat{a}^\dagger_{j_p\nu_p} \hat{a}_{\tilde{j_h\nu_h}}]_{j_t} || I_A > R_{\ell_p}(r_1) \\
& \quad \times\ (-)^{\eta_1} \hat{\ell}_h \hat{\ell}_c \hat{\eta}_1 (\ell_c 0\eta_1 0|\ell_p 0)\ W(\ell_c\eta_1\ell_1\ell_h; \ell_p\lambda_1) \\
& \quad \times\ \frac{2\pi}{\hat{\ell}_h^2} \sum_{m_1} \hat{\eta}_1 (\eta_1 0\lambda_1 m_1|\ell_h m_1) \int R_{\ell_h}(r'_1) Y_{\ell_h m_1}(\theta, 0) Y^*_{\lambda_1 m_1}(\theta', 0) d\mu'
\end{aligned}
$$

For a nucleon scattering, the projectile density is just replaced by

$$
\begin{aligned}
\rho^E_{P,\lambda_2\ell\ell_c}(r_b, r_1, r) &= \frac{2\pi}{\hat{\lambda}_2^2} \sum_m \hat{\ell}(\ell 0\ell_c m|\lambda_2 m)\hat{\lambda}_2(-)^{\lambda_2}(-\frac{1}{r_1 r_b r}) Y_{\lambda_2 m}(\mu') Y^*_{\ell_c m}(\mu) \\
\mu &= \frac{r_b^2 + r_1^2 - r^2}{2 r_1 r_b}, \quad \mu' = \frac{r_b - r_1\mu}{r}
\end{aligned}
$$

4 Details of Input

4.1 Relativistic kinematics

What we want to do is the Lorentz transformation of Lab system to c.m. system such that

$$(E_0 + m_T, k_0) \quad \rightarrow \quad (\omega, 0)$$

where $E_0 - m_P = E_{lab}$, and $E_0^2 = m_P^2 + k_0^2$. Note that E_{lab} denotes the kinetic energy of lab system. Thus we have

$$\begin{aligned}
\begin{pmatrix} \omega \\ 0 \end{pmatrix} &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_0 + m_T \\ k_0 \end{pmatrix} \\
\beta_{cm} &= \frac{\vec{k}_0}{E_0 + m_T} \quad \text{(Velocity of c.m. wrt lab frame)} \\
\gamma_{cm} &= \frac{1}{\sqrt{1-\beta^2}} = \frac{E_0 + m_T}{\sqrt{(E_0+m_T)^2 - k_0^2}} = \frac{E_0 + m_T}{\sqrt{s}} \\
s \; = \; \omega^2 &= (E_0 + m_T)^2 - k_0^2 \\
&= (E_{lab} + m_P + m_T)^2 - (E_{lab} + m_P)^2 + m_P^2 \\
&= (m_P + m_T)^2 + 2E_{lab} m_T
\end{aligned}$$

Thus the c.m. kinetic energy, E_{cm}, is simply

$$E_{cm} \;=\; \omega - (m_T + m_P) = \sqrt{s} - (m_T + m_P)$$

We now obtain the c.m. energy and wave numbers by Lorentz transformation of (E_0, k_0),

$$\begin{pmatrix} E \\ k \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_0 \\ k_0 \end{pmatrix} = \frac{1}{\sqrt{s}} \begin{pmatrix} s - m_T^2 - m_T E_0 \\ m_T k_0 \end{pmatrix}$$

since

$$\begin{aligned}
E &= \gamma E_0 - \gamma\beta k_0 = \frac{E_0 + m_T}{\sqrt{s}}(E_0 - \frac{k_0^2}{E_0 + m_T}) \\
&= \frac{1}{\sqrt{s}}[(E_0 + m_T)^2 - k_0^2 - m_T E_0 - m_T^2] = \frac{1}{\sqrt{s}}(s - m_T^2 - m_T E_0) \\
k_{cm} &= -\gamma\beta E_0 + \gamma k_0 = \frac{E_0 + m_T}{\sqrt{s}}(-\frac{E_0 k_0}{E_0 + m_T} + k_0) = \frac{1}{\sqrt{s}} m_T k_0
\end{aligned}$$

We thus have the wave number and the masses of target and projectile in the c.m. system, as follows.

$$\begin{aligned}
k_{cm}^2 &= \frac{m_T^2}{s} k_0^2 = \frac{m_T^2}{s}(E_0^2 - m_P^2) = \frac{m_T^2}{s}[(E_{lab} + m_P^2)^2 - m_P^2] = \frac{m_T^2}{s}(E_{lab}^2 + 2E_{lab} m_P) \\
m_T' &= \gamma_{cm} m_T = \frac{E_0 + m_T}{\sqrt{s}} = \frac{m_T}{\sqrt{s}}(E_{lab} + m_P + m_T) \\
m_P' &= (1 + \beta_0 \beta_{cm})\gamma_0 \gamma_{cm} m_P \qquad \text{with} \;\; \beta_0 = \frac{k_0}{E_0}, \;\; \gamma_0 = \frac{E_0}{m_P} \\
&= [1 + \frac{k_0^2}{E_0(E_0 + m_T)}] \frac{E_0(E_0 + m_T)}{\sqrt{s}} = \frac{1}{\sqrt{s}}[E_0^2 + E_0 m_T + k_0^2] = \frac{1}{\sqrt{s}}[2E_0^2 + E_0 m_T - m_P^2] \\
&= \frac{1}{\sqrt{s}}[2E_{lab}^2 + 4E_{lab} m_P + m_P(m_P + m_T) + E_{lab} m_T] \approx \frac{1}{\sqrt{s}}[m_P(m_P + m_T) + E_{lab} m_T]
\end{aligned}$$

with $m_T >> m_P$ or E_{lab}.

4.2 Love-Franey interaction

The Love-and Franey interactions[8] are defined as

$$\begin{aligned} V^C(r) &= \sum_{i=1}^{N_C} V_i^C\ Y(r/R_i), \qquad Y(x)=e^{-x}/x \\ V^{LS}(r) &= \sum_{i=1}^{N_{LS}} V_i^{LS}\ Y(r/R_i), \\ V^T(r) &= \sum_{i=1}^{N_T} V_i^T\ r^2 Y(r/R_i), \end{aligned}$$

4.3 Spectroscopic amplitudes in the projectile system

We calculate the spectroscopic amplitudes in the projectile system defined as, $\hat{s}_1^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a >$, which appears in the expansion coefficient $\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t}$ (SUBROUTINE AFACAL), [Eq.(13)],

$$\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t} = W(s_t\ell_t s_t \ell_1; j_t k)\hat{s}_t^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a >$$

The Wigner-Eckart theorem states that the the reduced matrix element is defined such that

$$\begin{aligned} < s_b m_b t_b \nu_b|[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}|s_a m_a t_a \nu_a > &= \hat{s}_b^{-1}(s_a m_a s_1 m_1|s_b m_b)(t_a \nu_a t_1 \tilde{\nu}_1|t_b \nu_b) < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a > \\ [c^\dagger c]_{s_1 t_1 \tilde{\nu}_1} &= \sum (s_a \sigma_1 s_b \sigma_2|s_1 m_1)(t_a \mu_1 t_b \mu_2|t_1 \tilde{\nu}_1)\hat{c}^\dagger_{\sigma_1 \mu_1}\hat{c}_{\sigma_2 \tilde{\mu}_2} \end{aligned}$$

1. Single nucleon state

The single-nucleon system can be written as

$$|s_a m_a t_a \nu_a > = c^\dagger_{m_a \nu_a}|0>$$

We thus have

$$\begin{aligned} I &\equiv < s_b m_b t_b \nu_b|[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}|s_a m_a t_a \nu_a > \\ &= \sum (s_a \sigma_1 s_b \sigma_2|s_1 m_1)(t_a \mu_1 t_b \mu_2|t_1 \tilde{\nu}_1) < 0|\hat{c}_{m_b \nu_b}\hat{c}^\dagger_{\sigma_1 \mu_1}\hat{c}_{\sigma_2 \tilde{\mu}_2}\hat{c}^\dagger_{m_a \nu_a}|0> \\ &= \sum (s_a \sigma_1 s_b \sigma_2|s_1 m_1)(t_a \mu_1 t_b \mu_2|t_1 \tilde{\nu}_1)\delta_{m_b,\sigma_1}(-)^{s_a+m_a}\delta_{m_a,-\sigma_2}\delta_{\nu_b,\mu_1}(-)^{t_a+\nu_a}\delta_{\nu_a,-\mu_2} \\ &= (-)^{s_a+m_a}(s_a m_b s_b, -m_a|s_1 m_1)(-)^{t_a+\nu_a+t_1+\nu_1}(t_a \nu_b t_b, -\mu_2|t_1, -\nu_1) \\ &= \hat{s}_1\hat{s}_b^{-1}(s_a m_a s_1 m_1|s_b m_b)\hat{t}_1\hat{t}_b^{-1}(t_a \nu_b t_1 \tilde{\nu}_1|t_b \nu_b) \end{aligned}$$

Comparing this equation with the Wigner-Eckart theorem gives

$$\hat{s}_1^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}||a >= \hat{t}_1^{-1}(\frac{1}{2}\nu_b \frac{1}{2}\tilde{\nu}_a|t_1 \tilde{\nu}_1)$$

Note that the spin part of the spectroscopic factor becomes unity for a single nucleon system.

[8] W. G. Love and M. A. Franey, Phys. Rev. **C15** 1396 (1977), **C24** 1073 (1981), **C27** 438(E) (1983), and **C31** 488 (1985).

2. Two-nucleon system

The two-nucleon system can be written as

$$|s_a m_a t_a \nu_a> \;=\; \frac{1}{\sqrt{2}}[c^\dagger c^\dagger]_{s_a m_a t_a \nu_a}|0>$$

We thus have

$$\begin{aligned}
I &= [c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}|s_a m_a t_a \nu_a> \\
&= \sum \frac{1}{\sqrt{2}}(\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2|s_1 m_1)(\frac{1}{2}\mu_1 \frac{1}{2}\mu_2|t_1\nu_1)(\frac{1}{2}\sigma_1' \frac{1}{2}\sigma_2'|s_a m_a)(\frac{1}{2}\mu_1' \frac{1}{2}\mu_2'|t_a\nu_a) \\
&\qquad \times \hat{c}^\dagger_{\sigma_1\mu_1}\hat{c}_{\sigma_2\tilde{\mu}_2}\hat{c}^\dagger_{\sigma_1\mu_1}\hat{c}^\dagger_{\sigma_2\mu_2}|0> \\
&= \frac{1}{\sqrt{2}}(1-(-)^{s_a+t_a})\sum(\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2|s_1 m_1)(-)^{1/2+\sigma_2}(\frac{1}{2},-\sigma_2 \frac{1}{2}\sigma_2'|s_a m_a) \\
&\qquad (\frac{1}{2}\mu_1 \frac{1}{2}\mu_2|t_1\nu_1)(-)^{1/2+\mu_2}(\frac{1}{2},-\mu_2 \frac{1}{2}\mu_2'|t_a\nu_a)\hat{c}^\dagger_{\sigma_1\mu_1}\hat{c}^\dagger_{\sigma_2\mu_2}|0> \\
J &= < s_b m_b t_b \nu_b|[c^\dagger c]_{s_1 t_1 \tilde{\nu}_1}|s_a m_a t_a \nu_a> \\
&= (1-(-)^{s_a+t_a})\sum(-)^{1/2+\sigma_2}(\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2|s_1 m_1)(\frac{1}{2},-\sigma_2 \frac{1}{2}\sigma_2'|s_a m_a)(\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2'|s_b m_b) \\
&\qquad (-)^{1/2+\mu_2}(\frac{1}{2}\mu_1 \frac{1}{2}\mu_2|t_1\nu_1)(\frac{1}{2},-\mu_2 \frac{1}{2}\mu_2'|t_a\nu_a)(\frac{1}{2}\mu_1 \frac{1}{2}\mu_2'|t_b\nu_b) \\
&= (1-(-)^{s_a+t_a})(s_a m_a s_1 m_1|s_b m_b)\hat{s}_1\hat{s}_a W(s_a\frac{1}{2}s_1\frac{1}{2};\frac{1}{2}s_b) \\
&\qquad \times\ (t_a\nu_a t_1\tilde{\nu}_1|t_b\nu_b)\hat{t}_1\hat{t}_a W(t_a\frac{1}{2}t_1\frac{1}{2};\frac{1}{2}t_b) \\
K &= \hat{s}_1^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1\tilde{\nu}_1}||a> \\
&= (1-(-)^{s_a+t_a})\hat{s}_a\hat{s}_b W(s_a\frac{1}{2}s_1\frac{1}{2};\frac{1}{2}s_b)(t_a\nu_a t_1\tilde{\nu}_1|t_b\nu_b)\hat{t}_a W(t_a\frac{1}{2}t_1\frac{1}{2};\frac{1}{2}t_b) \\
&\equiv L\times M \\
L &= (1-(-)^{s_a+t_a})\hat{s}_a\hat{s}_b W(s_a\frac{1}{2}s_1\frac{1}{2};\frac{1}{2}s_b)\hat{t}_a\hat{t}_b W(t_a\frac{1}{2}t_1\frac{1}{2};\frac{1}{2}t_b) \\
M &= \hat{t}_b^{-1}(t_a\nu_a t_1\tilde{\nu}_1|t_b\nu_b) \\
&= \hat{t}_b^{-1}(-)^{t_1+\nu_1}(t_a\nu_a t_1,-\nu_1|t_b\nu_b) \\
&= \hat{t}_b^{-1}(-)^{t_1+\nu_1+t_a-\nu_a}\hat{t}_b\hat{t}_1^{-1}(t_a\nu_a t_b,-\nu_b|t_1\nu_1) \\
&= (-)^{t_1+\nu_1+t_a-\nu_a}\hat{t}_1^{-1}(t_a\nu_a t_b,-\nu_b|t_1\nu_1)
\end{aligned}$$

The expansion coefficient $\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t}$, [Eq.(13)], becomes

$$\begin{aligned}
\alpha^{j_t s_t \nu_1}_{t_1 s_1 \ell_2 k \ell_t} &= W(s_t\ell_t s_t\ell_1;j_t k)\hat{s}_t^{-1}\hat{t}_1^{-1} < b||[c^\dagger c]_{s_1 t_1\tilde{\nu}_1}||a> \\
&= W(s_t\ell_t s_t\ell_1;j_t k)\times M\times L
\end{aligned}$$

For deuteron case, where $t_a=\nu_a=0$ and $s_a=1$, L becomes

$$\begin{aligned}
L &= 2\sqrt{3}\hat{s}_b W(1\frac{1}{2}s_1\frac{1}{2};\frac{1}{2}s_b)\hat{t}_b W(0\frac{1}{2}t_1\frac{1}{2};\frac{1}{2}t_b) \\
&= \sqrt{6}\hat{s}_b W(1s_1\frac{1}{2}\frac{1}{2};s_b\frac{1}{2})\delta(t_1,t_b)
\end{aligned}$$

4.4 Charge density distribution

1) Deuteron

a. Hulthen wavefunction

$$\begin{aligned}\phi_d &= \frac{1}{\sqrt{4\pi}}\frac{u(r)}{r} \\ u(r) &= \sqrt{\frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}}(e^{-\alpha r}-e^{-\beta r})\end{aligned}$$

where $\alpha^{-1} = 4.3$ fm, and $\beta = 7\alpha$.

b. Scattering wave (N. Austern, NP **7**, 195 (1958).)

$$\begin{aligned}\phi_d &= \frac{1}{kr}\sin\delta e^{i\delta}(\cot\delta\sin kr+\cos kr-e^{-\eta r}) \\ &= \frac{1}{kr}e^{i\delta}[\sin(kr+\delta)-\sin\delta e^{-\eta r}]\end{aligned}$$

where $k = \sqrt{\frac{2\mu E}{\hbar^2}} = k_{unit}\sqrt{\mu E} = k_{unit}\sqrt{0.5E}$.

2) ^{3}He

(C.W. de Jager, H. de Vries and C. de Vries, Atomic data and nuclear data tables, **14** 479 (1974), and **36** 495 (1987).)

a. Gaussian

$$\rho_0(r) = \frac{z}{8\pi^{3/2}}[\frac{1}{a^3}\exp(\frac{-r^2}{4a^2}) - \frac{c^2(6b^2-r^2)}{4b^7}\exp(\frac{-r^2}{4b^2})]$$

with $a = 0.675,\ b = 0.836,\ c = 0.366$ fm.

b. Sum of Gaussian's

A Properties of Spherical Harmonics

1) Orthogonality

$$\int d\hat{r} Y_{\ell_1 m_1}(\hat{r}) Y^*_{\ell_2 m_2}(\hat{r}) = \delta(\ell_1, \ell_2)\delta(m_1, m_2)$$

2) Conjugation

$$Y^*_{\ell m}(\theta, \phi) = (-)^m Y_{\ell, -m}(\theta, \phi)$$

3) Parity

$$Y_{\ell m}(-\hat{r}) = Y_{\ell m}(\pi - \theta, \pi + \phi) = (-)^\ell Y_{\ell m}(\hat{r})$$

4) Relation with Legendre function

$$\begin{aligned} P_\ell(\cos\theta) &= \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\hat{r}) Y^*_{\ell m}(\hat{r}'), \qquad \cos\theta = \hat{r}\cdot\hat{r}' \\ Y_{\ell 0}(\theta, \phi) &= \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \end{aligned}$$

5) Product

$$\begin{aligned} Y_{\ell_1 m_1}(\hat{r}) Y_{\ell_2 m_2}(\hat{r}) &= \sum_\ell d_{\ell_1 \ell_2 \ell} (\ell_1 m_1 \ell_2 m_2 | \ell m) Y_{\ell m}(\hat{r}) \\ d_{\ell_1 \ell_2 \ell} &\equiv [\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}]^{1/2} (\ell_1 0 \ell_2 0 | \ell 0) = \frac{\hat{\ell}_1 \hat{\ell}_2}{\sqrt{4\pi}\hat{\ell}} (\ell_1 0 \ell_2 0 | \ell 0) \\ \hat{\ell} &\equiv \sqrt{2\ell+1} \end{aligned}$$

6) Moshinsky Bracket

$$\begin{aligned} [Y_{\ell_1}(\hat{r}_1) Y_{\ell_2}(\hat{r}_2)]_{\ell m} &= \sum_{m_1 m_2} (\ell_1 m_1 \ell_2 m_2 | \ell m) Y_{\ell_1 m_1}(\hat{r}_1) Y_{\ell_2 m_2}(\hat{r}_2) \\ Y_{\ell_1 m_1}(\hat{r}_1) Y_{\ell_2 m_2}(\hat{r}_2) &= \sum_{\ell m} (\ell_1 m_1 \ell_2 m_2 | \ell m) [Y_{\ell_1}(\hat{r}_1) Y_{\ell_2}(\hat{r}_2)]_{\ell m} \end{aligned}$$

B Vector Coupling Coefficients

B.1 Clebsch-Gordan coefficients: Coupling of 2 angular momenta

1) Orthonormality

$$\begin{aligned} \sum_{m_1 m_2} (j_1 m_1 j_2 m_2 | jm)(j_1 m_1 j_2 m_2 | j' m') &= \delta(j, j')\delta(m, m') \\ \sum_{jm} (j_1 m_1 j_2 m_2 | jm)(j_1 m_1' j_2 m_2' | jm) &= \delta(m_1, m_1')\delta(m_2, m_2') \end{aligned}$$

2) Symmetries

$$\begin{aligned} (j_1 m_1 j_2 m_2 | jm) &= (-)^{j_1+j_2-j} (j_2 m_2 j_1 m_1 | jm) \\ &= (-)^{j_1+j_2-j} (j_1, -m_1 j_2, -m_2 | j, -m) \;=\; (j_2, -m_2 j_1, -m_1 | j, -m) \\ &= (-)^{j_1 - m_1} \frac{\hat{j}}{\hat{j}_2} (j_1, m_1 j, -m | j_2, -m_2) \;=\; (-)^{j_1 - m_1} \frac{\hat{j}}{\hat{j}_2} (jm j_1, -m_1 | j_2 m_2) \\ &= (-)^{j_2 + m_2} \frac{\hat{j}}{\hat{j}_1} (j, -m j_2 m_2 | j_1, -m_1) \end{aligned}$$

3) Analytic expression

$$
\begin{aligned}
(j_1 m_1 j_2 m_2 | j m_j) &= \delta_{m,m_1+m_2} \sqrt{\frac{(2j+1)(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}} \\
&\times \sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!} \\
&\times \sum_k [\frac{(-)^k}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!} \\
&\qquad \times \frac{1}{(j-j_2+m_1+k)!(j-j_1-m_2+k)!}]
\end{aligned}
$$

4) Special Values

$$
\begin{aligned}
(j_1 0 j_2 0 | j 0) &= (-)^{g+j} \hat{j} \sqrt{\frac{(j_1+j_2-j)!(j+j_1-j_2)!(j+j_2-j_1)!}{(2g+1)!}} \frac{g!}{(g-j)!(g-j_1)!(g-j_2)!} \\
&\quad \text{for } j+j_1+j_2 = 2g = \text{ even} \\
&= 0 \quad \text{for } j+j_1+j_2 = 2g = \text{ odd} \\
(j_1 m_1 0 0 | j m) &= \delta(j_1, j)\delta(m_1, m)
\end{aligned}
$$

B.2 Racah coefficients: Coupling of 3 angular momenta

1) Definition

$$
\begin{aligned}
& j_1 + j_2 = J_{12}, \quad j_2 + j_3 = J_{23} \quad J_1 2 + j_3 = j_1 + J_{23} = J \\
& < j_1 j_2 (J_{12}) j_3 J M | j_1, j_2 j_3 (J_{23}) J M >= \hat{J}_{12} \hat{J}_{23} W(j_1 j_2 J j_3; j_{12} J_{23}) \\
& = \sum_{m_1 m_2 m_3} (j_1 m_1 j_2 m_2 | J_{12} M_{12})(J_{12} M_{12} j_3 m_3 | J M)(j_2 m_2 j_3 m_3 | J_{23} M_{23})(j_1 m_1 J_{23} M_{23} | J M)
\end{aligned}
$$

In $W(abcd; ef)$, $(abe)(cde)(acf)(bdf)$ must satisfy the triangular relation ($|a-b| \le c \le a+b$).

2) Symmetries

$$
\begin{aligned}
W(abcd; ef) &= W(badc; ef) = W(cdab; ef) = W(acbd; fe) \\
&= (-)^{e+f-a-d} W(ebcf; ad) = (-)^{e+f-b-c} W(aefd; bc)
\end{aligned}
$$

3) Orthogonality

$$
\sum_e \hat{e}^2 W(abcd; ef) W(abcd; ef') = \delta(f, f')/\hat{f}^2
$$

$$
\sum_c \hat{c}^2 W(abcd; ef) W(ab'cd; ef) = \delta(b, b')/\hat{b}^2
$$

$$
\sum_d \hat{d}^2 W(abcd; ef) W(a'bcd; ef') = \delta(a, a')/\hat{a}^2
$$

4) Equality

$$\begin{aligned}
\sum_e (-)^{a+b-e}\hat{e}^2 W(abcd;ef)W(bacd;ef') &= W(aff'b;cd) \\
\sum_\lambda \hat{\lambda}^2 W(a'\lambda be;ac')W(c\lambda de';c'e)W(a'\lambda fc;ac') &= W(abcd;ef)W(a'bc'd;e'f) \\
\sum_\beta (a\alpha b\beta|e\epsilon)(e\epsilon d\delta|c\gamma)(b\beta d\delta|f\varphi) &= \hat{e}\hat{f}(a\alpha f\varphi|c\gamma)W(abcd;ef) \\
\sum_f \hat{e}\hat{f}(a\alpha f\varphi|c\gamma)(b\beta d\delta|f\varphi)W(abcd;ef) &= (a\alpha b\beta|e\epsilon)(e\epsilon d\delta|c\gamma) \\
\sum_e \hat{e}\hat{f}(a\alpha b\beta|e\epsilon)(e\epsilon d\delta|c\gamma)W(abcd;ef) &= (a\alpha f\varphi|c\gamma)(c\gamma d\delta|f\varphi)
\end{aligned}$$

5) Special values

$$\begin{aligned}
W(0bcd;bc) &= \hat{b}^{-1}\hat{c}^{-1} \\
W(aa'cc';0f) &= (-)^{a+c-f}\hat{a}^{-1}\hat{c}^{-1}\delta(a,a')\delta(c,c')
\end{aligned}$$

B.3 X (Fano) coefficients and 9-j (Wigner) symbols: Coupling of 4 angular momenta

1) Definition

$$\begin{aligned}
j_1+j_2=J_{12},\quad j_3+j_4=J_{34}, \qquad & j_1+j_3=J_{13},\quad j_2+j_4=J_{24}, \\
J_{12}+J_{34} &= J_{13}+J_{24}=J \\
< j_1j_2(J_{12})j_3j_4(J_{34})JM|j_1j_3(J_{13})j_2j_4(J_{24})JM > &= X(j_1j_2J_{12},j_3j_4J_{34};J_{13}J_{24}J) \\
&= \hat{J}_{12}\hat{J}_{34}\hat{J}_{13}\hat{J}_{24}U\begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ j_{13} & j_{24} & J \end{pmatrix}
\end{aligned}$$

2) Relation with Racah coefficients

$$\begin{aligned}
U\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} &= (-)^\sigma \sum_\lambda \hat{\lambda}^2 W(bcef;\lambda a)W(bcf'e';\lambda d)W(efe'f';\lambda g) \\
\sigma &= a+b+c+d+e+e'+f+f'+g = \text{ integer}
\end{aligned}$$

3) Symmetries

$$\begin{aligned}
U\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} &= U\begin{pmatrix} a & c & f \\ b & d & f' \\ e & e' & g \end{pmatrix} \quad \text{(transposition)} \\
&= (-)^\sigma U\begin{pmatrix} c & d & e' \\ a & b & e \\ f & f' & g \end{pmatrix} = (-)^\sigma U\begin{pmatrix} f & f' & g \\ c & d & e' \\ a & b & e \end{pmatrix}
\end{aligned}$$

4) Special values

$$\begin{aligned}
U\begin{pmatrix} a & b & e \\ a & b & e \\ f & f' & g \end{pmatrix} &= 0 \quad (f+f'+g = \text{ odd}) \\
U\begin{pmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{pmatrix} &= (-)^{e+f-a-d}\hat{e}^{-1}\hat{f}^{-1}W(abcd;ef)
\end{aligned}$$

C Multipole Expansions

C.1 Scalar function of r_{12}

We wish to express a scalar function $f(r_{12})$ in terms of function of r_1 and r_2, where

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2, \quad r_{12}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \mu, \quad \mu = \cos\theta$$

where θ is the angle between $\vec{r}_1$ and $\vec{r}_2$, as shown in Fig.10.

It can simply be done by Legendre expansion,

$$\begin{aligned} f(r_{12}) &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)P_\ell(\cos\theta) \\ f_\ell(r_1, r_2) &= \frac{1}{2}\int_{-1}^{1} f(r_{12})P_\ell(\cos\theta)d\mu \end{aligned}$$

We now express $f(r_{12})$ in the form of

$$\begin{aligned} f(r_{12}) &= \sum_\ell f'_\ell(r_1, r_2)(-)^\ell [Y_\ell Y_\ell]_{00} \\ f'_\ell(r_1, r_2) &= \sqrt{16\pi^3}\int_{-1}^{1} f(r_{12})Y_{\ell 0}(\theta, 0)d\mu, \qquad \mu \equiv \cos\theta \end{aligned}$$

Proof:

$$\begin{aligned} f(r_{12}) &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)P_\ell(\cos\theta) \\ &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)\frac{4\pi}{(2\ell+1)}\sum_m Y_{\ell m}(\hat{r}_1)Y^*_{\ell m}(\hat{r}_2) \\ &= \sum_\ell f_\ell(r_1, r_2)4\pi(-)^\ell \hat{\ell}(\ell m \ell, -m|00)[Y_\ell Y_\ell]_{00} \\ &= \sum_\ell f_\ell(r_1, r_2)4\pi(-)^\ell \hat{\ell}[Y_\ell Y_\ell]_{00} \end{aligned}$$

Thus we have

$$\begin{aligned} f'_\ell(r_1, r_2) &= f_\ell(r_1, r_2)4\pi\hat{\ell} \\ &= 4\pi\hat{\ell}\frac{1}{2}\int_{-1}^{1} f(r_{12})P_\ell(\cos\theta)d\mu \\ &= 2\pi\hat{\ell}\int_{-1}^{1} f(r_{12})P_\ell(\mu)d\mu \\ &= 2\pi\hat{\ell}\int_{-1}^{1} f(r_{12})\sqrt{4\pi}\hat{\ell}^{-1}Y_{\ell 0}(\theta, 0)d\mu \\ &= \sqrt{16\pi^3}\int_{-1}^{1} f(r_{12})Y_{\ell 0}(\theta, 0)d\mu, \qquad \text{QED} \end{aligned}$$

C.2 Vector function of $\vec{r}_{12}$

A vector function $f(r_{12})Y_{\ell m}(\hat{r}_{12})$ can be expanded

$$
\begin{aligned}
f(r_{12})Y_{\ell m}(\hat{r}_{12}) &= \sqrt{4\pi}\sum_{\ell_1\ell_2} f_{\ell\ell_1\ell_2}(r_1,r_2)[Y_{\ell_1}Y_{\ell_2}]_{\ell m} \\
f_{\ell\ell_1\ell_2}(r_1,r_2) &= \frac{1}{\sqrt{4\pi}}\sum_m \int f(r_{12})Y_{\ell m}(\hat{r}_{12})[Y_{\ell_1}Y_{\ell_2}]^*_{\ell m} d\hat{r}_1 d\hat{r}_2 \\
&= \frac{1}{\sqrt{4\pi}}\frac{1}{2\ell+1}\sum_m \int\int f(r_{12})Y_{\ell m}(\hat{r}_{12}) \\
&\qquad \times \sum_{m_1m_2}(\ell_1 m_1 \ell_2 m_2|\ell m)Y_{\ell_1 m_1}(\hat{r}_1)Y^*_{\ell_2 m_2}(\hat{r}_2)d\hat{r}_1 d\hat{r}_2
\end{aligned}
$$

Now we choose $\hat{r}_1 \parallel \hat{z}$, and thus $m_1 = 0$ and $m_2 = m$. Then we have

$$
Y_{\ell_1 m_1}(\hat{r}_1) = Y_{\ell_1 0}(\cos\theta) = \frac{\hat{\ell}_1}{\sqrt{4\pi}}, \qquad Y_{\ell_2 m_2}(\hat{r}_2) = Y_{\ell_2 m}(\theta,0), \qquad Y_{\ell m}(\hat{r}_{12}) = Y_{\ell m}(\theta',0)
$$

$$
\int d\hat{r}_1 \;\Rightarrow\; 4\pi, \qquad \int d\hat{r}_2 \;\Rightarrow\; 2\pi\int d\mu
$$

where θ is the angle between z-axis ($\vec{r}_1$) and $\vec{r}_2$ while θ' the angle between z-axis and $\vec{r}_{12}$ as shown in Fig.11. Then we have

$$
\begin{aligned}
f_{\ell\ell_1\ell_2}(r_1,r_2) &= \frac{1}{2\ell+1}\sum_m \int f(r_{12})Y_{\ell m}(\theta',0)\;\frac{1}{\sqrt{4\pi}}\frac{\hat{\ell}_1}{\sqrt{4\pi}}(\ell_1 0 \ell_2 m|\ell m)4\pi Y^*_{\ell_2 m}(\theta,0)2\pi d\mu \\
&= \frac{2\pi}{2\ell+1}\sum_m \hat{\ell}_1(\ell_1 0 \ell_2 m|\ell m)\int f(r_{12})Y_{\ell m}(\theta',0)Y^*_{\ell_2 m}(\theta,0)d\mu
\end{aligned}
$$

where

$$
\begin{aligned}
&\mu \equiv \cos\theta, \quad (\hat{r}_1(\parallel \hat{k})\cdot\hat{r}_2) && r_{12}^2 = r_1^2 + r_2^2 - 2r_1r_2\mu \\
&\mu' \equiv \cos\theta' = \frac{r_2\mu - r_1}{r_{12}} \quad (\hat{r}_1(\parallel \hat{k})\cdot\hat{r}_{12}), && r_2^2 = r_{12}^2 + r_1^2 + 2r_1r_{12}\mu'
\end{aligned}
$$

C.3 Solid harmonics

The solid harmonics is defined as

$$
\mathcal{Y}_{\ell m}(r) = r^\ell Y_{\ell m}(\hat{r})
$$

When $\vec{r} = s_a\vec{r}_a + t_a\vec{r}_b$, this solid harmonics can be expressed in terms of $\vec{r}_a$ and $\vec{r}_b$ as

$$
\mathcal{Y}_{\ell m}(r) = \sum_{\lambda_1\lambda_2}\sqrt{4\pi}\left[\frac{(2\ell+1)!}{(2\lambda_2+1)!(2\lambda_1+1)!}\right]^{1/2}\delta_{\lambda_1+\lambda_2,\ell}\;(s_ar_a)^{\lambda_1}(t_ar_b)^{\lambda_2}[Y_{\lambda_1}(\hat{r}_a)Y_{\lambda_2}(\hat{r}_b)]_{\ell m}
$$

See "M. Moshinsky, NP **13**, 104 (1959)" for the proof.

D Time Reversal State

The time reversal state is defined as

$$\mathcal{T}\phi_{\ell m} \equiv \phi_{\tilde{\ell m}} \equiv (-)^{\ell+m}\phi_{\ell,-m}$$

where $\mathcal{T}$ is the time reversal operator. The CG coefficient with a time reversal state can be written

$$(\ell_1 m_1 \ell_2 m_2|\tilde{\ell m}) = (-)^{\ell+m}(\ell_1 m_1 \ell_2 m_2|\ell, -m)$$